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Gauge-invariant correlation functions in light-cone superspace

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Abstract

We initiate a study of correlation functions of gauge-invariant operators in $\mathcal{N} = 4$ super Yang-Mills theory using the light-cone superspace formalism. Our primary aim is to develop efficient methods to compute perturbative corrections to correlation functions. This analysis also allows us to examine potential subtleties which may arise when calculating off-shell quantities in light-cone gauge. We comment on the intriguing possibility that the manifest $\mathcal{N} = 4$ supersymmetry in this approach may allow for a compact description of entire multiplets and their correlation functions.

1 Introduction

Correlation functions of gauge-invariant operators are the fundamental observables in any conformally invariant gauge theory. In the case of the $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory the calculation of such correlation functions has received significant attention in recent years because of the central role they play in the context of the AdS/CFT correspondence [1–3].

In this paper we initiate a systematic study of correlation functions of gauge-invariant composite operators in $\mathcal{N} = 4$ SYM using the formalism of light-cone superspace. The unique characteristic of this formalism is that it provides a description, solely in terms of physical degrees of freedom, in which the full $\mathcal{N} = 4$ supersymmetry as well as the SU(4) R-symmetry are manifestly realised. This is achieved at the expense of manifest Lorentz invariance.

The formulation of $\mathcal{N} = 4$ SYM in light-cone superspace was introduced in [4, 5] and proved to be a powerful tool [5, 6] for the proof of the ultra-violet finiteness of the theory to all orders in perturbation theory (see also [7]). The formalism was more recently adapted to deformations of the $\mathcal{N} = 4$ SYM theory with less or no supersymmetry [8–10] and used to prove the absence of ultraviolet divergences in these models [11].

While the light-cone superspace formalism has a number of remarkable properties, its application to the perturbative computation of physical observables has been very limited. The focus of the present paper is on introducing the main features of $\mathcal{N} = 4$ light-cone superspace as applied to the study of gauge-invariant correlation functions in position space. To this end we present the one-loop calculation of a simple four-point correlator of gauge-invariant scalar operators belonging to the super-multiplet of the energy-momentum tensor. Specifically, we will reproduce the known tree-level and one-loop results for a four point function of composite operators which are bilinear in the elementary scalars and transform in the $\mathbf{20}'$ representation of the SU(4) R-symmetry of $\mathcal{N} = 4$ SYM.

Conformal symmetry constrains the form of correlation functions and supersymmetry further restricts the quantum corrections they receive. This is especially the case in $\mathcal{N} = 4$ SYM, which has the maximal amount of rigid supersymmetry in four dimensions. Therefore a formalism with manifest $\mathcal{N} = 4$ supersymmetry is particularly desirable and it is natural to expect that it will prove very efficient for the calculation of correlation functions.

Another benefit of manifest $\mathcal{N} = 4$ supersymmetry is the possibility of describing an entire multiplet of operators in terms of a single super-operator. Understanding how powerful the light-cone superspace formulation really is in this respect will require further investigation of the role played by the non-linearly realised dynamical supersymmetries, but we will make some comments about this point in the following.

The study of gauge invariant correlation functions also allows us to address general questions concerning the consistency of the light-cone gauge – and specifically its superspace implementation – for the calculation of off-shell quantities. This is an important issue in general and particularly so for a conformally invariant theory such as $\mathcal{N} = 4$ SYM. In this case it is necessary to verify that no spurious infra-red divergences are present in gauge-invariant observables. Such spurious infra-red divergences could potentially be introduced by the elimination of the non-physical degrees of freedom through the formal solution of the equations of motion. Integrating out the unphysical fields from the action leads to the appearance of $1/\partial_-$ operators, which in turn could potentially introduce infra-red singularities in Feynman diagrams. While the present paper is only a first step towards a systematic understanding of these issues, we will explicitly show that unphysical infra-red divergences are absent

from tree-level and one-loop corrections to a particular gauge-invariant four-point correlation function thanks to non-trivial cancellations.

This paper is organised as follows. In the next section we briefly review the essential ingredients of the light-cone superspace formulation of $\mathcal{N} = 4$ SYM and we present the superfield propagator and Feynman rules in position space. In section 3 we discuss gauge-invariant super-operators and some aspects of their correlation functions. Section 4 contains the calculation of a four point correlator at tree-level and one-loop. Future directions and open problems are discussed in the concluding section and various details of the calculations are presented in several appendices.

2 $\mathcal{N} = 4$ light cone superspace

In this section we briefly review the formulation of $\mathcal{N} = 4$ SYM in light-cone superspace [4, 5]. We limit ourselves to aspects of the formalism which are relevant for the subsequent analysis and refer the reader to the original papers and to more comprehensive reviews presented in [11, 12] for further details.

2.1 Light-cone superspace formulation of $\mathcal{N} = 4$ SYM

The field content of the $\mathcal{N} = 4$ SYM theory comprises a gauge field, A_μ , four Weyl fermions, ψ_α^m , and their conjugates, $\bar{\psi}_{m\dot{\alpha}}$, $m = 1, \dots, 4$, and six real scalars, φ^i , $i = 1, \dots, 6$. All fields transform in the adjoint representation of the gauge group, which in the following will be taken to be $SU(N)$. The theory possesses a $SU(4)$ R-symmetry, hereby referred to as $SU(4)_R$, under which the gauge field is a singlet, the fermions transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ and the scalars in the $\mathbf{6}$.

The light-cone gauge description of the theory eliminates unphysical degrees of freedom. The A_- component of the gauge field is set to zero (gauge choice) and the A_+ component is integrated out, leaving two transverse components, A and \bar{A} . Similarly half of the components of the four Weyl fermions, ψ_α^m , and their conjugates, $\bar{\psi}_{m\dot{\alpha}}$, are integrated out after light-cone projection, leaving four one-component fermionic fields, λ^m , and their conjugates, $\bar{\lambda}_m$. The $\mathcal{N} = 4$ multiplet is completed by the six real scalar fields, φ^i , $i = 1, \dots, 6$, which can be written as $SU(4)_R$ bi-spinors, φ^{mn} , $m, n = 1, \dots, 4$, satisfying the reality condition

$$\bar{\varphi}_{mn} \equiv (\varphi^{mn})^* = \frac{1}{2} \varepsilon_{mnpq} \varphi^{pq}. \quad (2.1)$$

The relation between the two parametrisations of the scalars is discussed in appendix A, which also summarises further details of the light-cone projection.

The $\mathcal{N} = 4$ light-cone superspace is constructed combining the four bosonic coordinates (x^+, x^-, x, \bar{x}) defined in (A.1) with the eight fermionic coordinates, θ^m and $\bar{\theta}_m$, $m = 1, \dots, 4$, transforming in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)_R$. In the following we will collectively denote the superspace coordinates by $z = (x^+, x^-, x, \bar{x}, \theta^m, \bar{\theta}_m)$. The full $\mathcal{N} = 4$ supersymmetry is manifest, with half of the supercharges (denoted by q^m and \bar{q}_m , referred to as kinematical) realised linearly as translations in the fermionic coordinates and the other half (referred to as dynamical) non-linearly realised [4, 13, 14].

A central role in the light-cone superspace formulation of $\mathcal{N} = 4$ SYM is played by the chiral derivatives, d^m and \bar{d}_m , defined as

$$d^m = -\frac{\partial}{\partial \bar{\theta}_m} + \frac{i}{\sqrt{2}} \theta^m \partial_-, \quad \bar{d}_m = \frac{\partial}{\partial \theta^m} - \frac{i}{\sqrt{2}} \bar{\theta}_m \partial_-, \quad m = 1, \dots, 4. \quad (2.2)$$

They obey

$$\{d^m, \bar{d}_n\} = i\sqrt{2}\delta_n^m \partial_- \quad (2.3)$$

and anti-commute with the supercharges q^m and \bar{q}_m .

The component fields in the light-cone $\mathcal{N} = 4$ multiplet can be packaged into a single complex superfield, $\Phi(x, \theta, \bar{\theta})$. This superfield is a $SU(4)_R$ singlet defined by the constraints [4, 6]

$$d^m \Phi(x, \theta, \bar{\theta}) = 0, \quad \bar{d}_m \bar{d}_n \Phi(x, \theta, \bar{\theta}) = \frac{1}{2} \varepsilon_{mnpq} d^p d^q \bar{\Phi}(x, \theta, \bar{\theta}), \quad (2.4)$$

where $\bar{\Phi} = \Phi^*$ satisfies $\bar{d}_m \bar{\Phi}(x, \theta, \bar{\theta}) = 0$. The unique solution to these constraints is a superfield with the following component expansion [4]

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = & -\frac{1}{\partial_-} A(y) - \frac{i}{\partial_-} \theta^m \bar{\lambda}_m(y) + \frac{i}{\sqrt{2}} \theta^m \theta^n \bar{\varphi}_{mn}(y) \\ & + \frac{\sqrt{2}}{6} \theta^m \theta^n \theta^p \varepsilon_{mnpq} \lambda^q(y) - \frac{1}{12} \theta^m \theta^n \theta^p \theta^q \varepsilon_{mnpq} \partial_- \bar{A}(y), \end{aligned} \quad (2.5)$$

where the chiral coordinate

$$y = (x^+, y^- = x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m, x, \bar{x}) \quad (2.6)$$

and the right hand side is understood to be a power expansion about x^- .

The superfields Φ and $\bar{\Phi}$, just like the component fields in the $\mathcal{N} = 4$ multiplet, transform in the adjoint representation of the gauge group $SU(N)$. They can therefore be represented as matrices,

$$\Phi(x, \theta, \bar{\theta}) = \Phi^a(x, \theta, \bar{\theta}) T^a, \quad \bar{\Phi}(x, \theta, \bar{\theta}) = \bar{\Phi}^a(x, \theta, \bar{\theta}) T^a, \quad (2.7)$$

where T^a , $a = 1, \dots, N^2 - 1$, are generators of the fundamental representation of $SU(N)$, satisfying

$$\text{Tr} \left(T^a T^b \right) = \frac{1}{2} \delta^{ab}. \quad (2.8)$$

The second constraint relation in (2.4) can be used to express the conjugate superfield, $\bar{\Phi}(x, \theta, \bar{\theta})$, in terms of $\Phi(x, \theta, \bar{\theta})$ as

$$\bar{\Phi}(x, \theta, \bar{\theta}) = \frac{1}{48} \frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi(x, \theta, \bar{\theta}), \quad (2.9)$$

where $\langle \bar{d}^4 \rangle = \varepsilon^{mnpq} \bar{d}_m \bar{d}_n \bar{d}_p \bar{d}_q$.

Using (2.9) the $\mathcal{N} = 4$ action in light-cone superspace can be written purely in terms of the superfield Φ as

$$\begin{aligned} \mathcal{S} = & \int d^4x d^4\theta d^4\bar{\theta} \left\{ \frac{1}{2} \Phi^a \left(-3 \frac{\langle \bar{d}^4 \rangle \square}{\partial_-^4} \right) \Phi_a \right. \\ & - 2g f^{abc} \left[\left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^3} \Phi_a \right) \Phi_b \partial_- \Phi_c + \frac{1}{48} \left(\frac{1}{\partial_-} \Phi_a \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_b \right) \partial_- \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_c \right) \right] \\ & \left. - \frac{g^2}{32} f^{eab} f^{ecd} \left[\frac{1}{\partial_-} (\Phi_a \partial_- \Phi_b) \frac{1}{\partial_-} \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_c \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-} \Phi_d \right) + \frac{1}{2} \Phi_a \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_b \right) \Phi_c \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_d \right) \right] \right\}, \end{aligned} \quad (2.10)$$

where $a, b, c, d, e, \dots = 1, \dots, N^2 - 1$ denote colour indices.

Here and in the following it is understood that we use the prescription of [5] for the $\frac{1}{\partial_-}$ operator. We will comment on potential subtleties associated with the presence of the $\frac{1}{\partial_-}$ factors in the discussion section.

2.2 Perturbative calculations in position space

Gauge-invariant correlation functions in a conformal field theory are most naturally studied in position space rather than momentum space. We now discuss some general aspects of perturbative calculations in configuration space using the formalism of light-cone superspace. We present the form of the superfield propagator and summarise the Feynman rules in position space. More details are provided in the appendices.

The superfield propagator in position space can be obtained inverting the kinetic operator in (2.10). We start by defining the generating functional, $Z[J]$, which gives rise to the Green functions of the $\mathcal{N} = 4$ superfield upon functional differentiation with respect to the chiral sources, $J(x, \theta, \bar{\theta})$. The Euclidean generating functional, with the coupling to external sources suitable for chiral superfields, is

$$Z[J] = \frac{\int [d\Phi] \exp\left(-\mathcal{S}[\Phi] + \int d^{12}z \Phi^a(z) \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a(z)\right)}{\int [d\Phi] \exp(-\mathcal{S}[\Phi])}, \quad (2.11)$$

where $z = (x, \theta, \bar{\theta})$ and $d^{12}z = d^4x d^4\theta d^4\bar{\theta}$. Because of the chirality constraint that both Φ and the sources must satisfy, the rules for functional differentiation in superspace involve subtleties which are addressed in appendix B.1.

In order to construct the super-propagator it is sufficient to focus on the free theory generating functional, $Z_0[J]$, obtained replacing $\mathcal{S}[\Phi]$ in (2.11) by the free action $\mathcal{S}_0[\Phi]$. We can write this generating functional as

$$Z_0[J] = \frac{\int [d\Phi] e^{-\frac{1}{2}(\Phi^a, \mathcal{K}_a^b \Phi_b) + \left(\Phi^a, \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a\right)}}{\int [d\Phi] e^{-\mathcal{S}[\Phi]}}, \quad (2.12)$$

where the inner products in the exponent are

$$-\frac{1}{2}(\Phi^a, \mathcal{K}_a^b \Phi_b) + \left(\Phi^a, \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a\right) = -\frac{1}{2} \int d^{12}z \Phi^a(z) \left(\mathcal{K}_a^b \Phi_b\right)(z) + \int d^{12}z \Phi^a(z) \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a(z). \quad (2.13)$$

Computing the Gaussian integral (2.12) we get

$$Z_0[J] = \exp\left\{\frac{1}{2} \left(\tilde{J}^a, [\mathcal{K}^{-1}]_a^b \tilde{J}_b\right)\right\} = \exp\left\{\frac{1}{2} \int d^{12}z d^{12}z' \tilde{J}^a(z) [\Delta(z - z')]_a^b \tilde{J}_b(z')\right\} \quad (2.14)$$

where

$$\tilde{J}^a(z) = \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J^a(z) \quad (2.15)$$

and the kernel, $\Delta(z - z')$, of the inverse kinetic operator, \mathcal{K}^{-1} , is the superfield propagator. The explicit form of $\Delta(z - z')$ is

$$\Delta_b^a(z - z') = -\frac{2}{(4!)^3} \frac{\delta_b^a}{(2\pi)^2} \frac{1}{(x - x')^2} \langle d^4 \rangle \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}'). \quad (2.16)$$

This result is derived in detail in appendix B.1. In appendix B.2 we show that (2.16) gives rise to the correct propagators for the component fields.

Notice that the superfield propagator in position space has essentially the same form as in momentum space [6], apart from the obvious replacement of $1/k^2$ by $1/(x - x')^2$. As a consequence the basic manipulations employed in the calculation of position space super Feynman diagrams are also the same used in momentum space. This represents a distinct feature compared to covariant superspace formalisms, where there are more significant differences between position and momentum space formulations.

The superfield interaction vertices in configuration space can be immediately read off from the superspace action (2.10). They involve a combination of chiral and space-time derivatives and $1/\partial_-$ operators acting on the various legs as well as group theory factors. The two cubic vertices are

$$\int d^{12}z (-2g) f^{abc} \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^3} \Phi_a \right) \Phi_b \bar{\partial} \Phi_c \longrightarrow (-2g) f^{abc} \begin{array}{c} \frac{\langle \bar{d}^4 \rangle}{\partial_-^3} \\ \text{---} a \text{---} \bullet \text{---} b \\ \text{---} z \text{---} \bar{\partial} \\ \text{---} c \end{array} \quad (2.17)$$

and

$$\int d^{12}z \left(-\frac{g}{24} \right) f^{abc} \left(\frac{1}{\partial_-} \Phi_a \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_b \right) \partial \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_c \right) \longrightarrow \left(-\frac{g}{24} \right) f^{abc} \begin{array}{c} \frac{1}{\partial_-} \\ \text{---} a \text{---} \bullet \text{---} b \\ \text{---} z \text{---} \frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \\ \text{---} \partial \frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \text{---} c \end{array} \quad (2.18)$$

Here we use a black dot to denote interaction vertices, which are integrated over the whole superspace, $z = (x, \theta, \bar{\theta})$, reflecting the fact that all intermediate steps in the calculations are manifestly $\mathcal{N} = 4$ supersymmetric. Notice that the two vertices (2.17) and (2.18) are complex conjugates of each other, although this is not apparent after elimination of the $\bar{\Phi}$ superfield. In the following we will refer to (2.17) and (2.18) as Vertex 3-I and Vertex 3-II respectively.

The two quartic vertices are

$$\int d^{12}z \left(-\frac{g^2}{32} \right) f^{eab} f^{ecd} \left[\frac{1}{\partial_-} (\Phi_a \partial_- \Phi_b) \frac{1}{\partial_-} \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_c \frac{\langle \bar{d}^4 \rangle}{\partial_-} \Phi_d \right) \right] \longrightarrow \left(-\frac{g^2}{32} \right) f^{eab} f^{ecd} \begin{array}{c} \frac{1}{\partial_-} \quad \frac{1}{\partial_-} \\ \text{---} a \text{---} \text{---} c \\ \text{---} b \text{---} \text{---} d \\ \text{---} \partial_- \quad \frac{\langle \bar{d}^4 \rangle}{\partial_-} \end{array} \quad (2.19)$$

and

$$\begin{aligned}
& \int d^{12}z \left(-\frac{g^2}{64} \right) f^{eab} f^{ecd} \left[\Phi_a \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_b \right) \Phi_c \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \Phi_d \right) \right] \\
& \longrightarrow \left(-\frac{g^2}{64} \right) f^{eab} f^{ecd} \quad \begin{array}{c} a \quad c \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ b \quad d \\ \frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \quad \frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \end{array} \quad (2.20)
\end{aligned}$$

In the vertex (2.19) the two $1/\partial_-$ operators in the shaded ovals act on both the adjacent legs. We will refer to (2.19) as Vertex 4-I and to (2.20) as Vertex 4-II.

Super Feynman diagrams constructed from these vertices and the propagator (2.16) contain space-time derivatives (∂ , $\bar{\partial}$ and ∂_- , but not ∂_+) as well as chiral derivatives d^m and \bar{d}_m defined in (2.2). All these derivatives can be integrated by parts in superspace integrals. They can also be transferred from one end point to the other of the super-propagator they act on, $\Delta(z - z')$, using the fact that the latter is only a function of the difference $(z - z')$. Moreover, the $1/\partial_-$ operators can effectively be “integrated by parts” as explained in (C.1).

The general strategy for the evaluation of position space Feynman diagrams is similar to that used in other superspace formulations. The first step consists in computing Grassmann integrals, utilising the fermionic δ -functions in the super-propagator. For this purpose one needs to free up one internal line of all the chiral derivatives, using repeated integrations by parts, and then use the relation (C.2) in appendix C.

Once the fermionic integrals at each interaction vertex have been computed, the external super-operators are projected onto specific components, thus drastically reducing the number of non-zero contributions.

At this point the resulting bosonic integrals can be directly compared to the corresponding expressions obtained using Lorentz covariant formulations. In section 4 we illustrate these steps in the case of a simple four-point function and we show how the light-cone superspace analysis reproduces the known covariant results prior to the evaluation of the final bosonic integrals.

3 Composite operators and correlation functions

Gauge-invariant operators in the $\mathcal{N} = 4$ SYM theory may be classified according to their transformation properties under the $\text{PSU}(2,2|4)$ superconformal symmetry group. They can be divided into two classes, protected operators belonging to short BPS multiplets of the superconformal group and unprotected ones belonging to generic long representations of $\text{PSU}(2,2|4)$. BPS operators have been classified and are characterised by shortening conditions expressed as relations among their $\text{PSU}(2,2|4)$ quantum numbers [15]. Their correlation functions have special properties and satisfy certain non-renormalisation theorems.

In this paper we will only consider examples of correlators of protected operators. Moreover we will confine ourselves to operators constructed from the elementary scalars in the $\mathcal{N} = 4$ multiplet, φ^{mn} . This ensures that the explicit form of the operators remain the same (in light-cone gauge) as in Lorentz covariant formulations. The simplest such operators

are scalars of dimension 2 belonging to the super-multiplet of the energy-momentum tensor, which is a short 1/2 BPS multiplet¹. They transform in the representation $\mathbf{20}'$ of the $SU(4)_R$ R-symmetry group and, in terms of the φ^{mn} representation for the elementary scalars, they take the form

$$\begin{aligned} Q^{[mn][pq]} &= \text{Tr}(\varphi^{mn}\varphi^{pq}) - \frac{1}{12}\varepsilon^{mnpq}\text{Tr}(\bar{\varphi}_{rs}\varphi^{rs}) \\ &= \frac{1}{3}\text{Tr}(2\varphi^{mn}\varphi^{pq} + \varphi^{mp}\varphi^{nq} - \varphi^{mq}\varphi^{np}) . \end{aligned} \quad (3.1)$$

We can express the same operators in terms of the representation φ^i of the scalars as $SU(4)_R$ vectors as

$$Q^{ij} = \text{Tr}(\varphi^i\varphi^j) - \frac{1}{6}\delta^{ij}\text{Tr}(\varphi^k\varphi^k) . \quad (3.2)$$

The equivalence of the two forms (3.1) and (3.2) can be verified using the identity (A.9).

In order to describe the operators (3.1)-(3.2) in light-cone superspace we introduce composite superfield operators which contain them in their component expansion. For this purpose it is convenient to work with the form (3.2) which, using (A.9) we can rewrite as

$$Q^{ij} = \frac{1}{8}\left(\sigma^{ipq}\sigma^{jrs} - \frac{1}{3}\delta^{ij}\varepsilon^{pqrs}\right)\text{Tr}(\bar{\varphi}_{pq}\bar{\varphi}_{rs}) . \quad (3.3)$$

From the form of the $\mathcal{N} = 4$ superfield (2.5) and the definition (2.2) of the chiral derivatives, \bar{d}_m , it is easy to verify that the scalar field $\bar{\varphi}_{mn}(x)$ in the expansion of $\Phi(z)$ can be isolated as follows

$$\bar{\varphi}_{mn}(x) = \frac{i}{\sqrt{2}}\left[\bar{d}_m\bar{d}_n\Phi(x, \theta, \bar{\theta})\right]_{\theta=\bar{\theta}=0} . \quad (3.4)$$

We can then define the super-operator

$$\mathcal{Q}^{ij}(z) = -\frac{1}{16}\left(\sigma^{ipq}\sigma^{jrs} - \frac{1}{3}\delta^{ij}\varepsilon^{pqrs}\right)\text{Tr}\left[(\bar{d}_p\bar{d}_q\Phi(z))(\bar{d}_r\bar{d}_s\Phi(z))\right] , \quad (3.5)$$

which contains (3.3) as its $\theta = \bar{\theta} = 0$ component,

$$Q^{ij}(x) = [\mathcal{Q}^{ij}(z)]_{\theta=\bar{\theta}=0} . \quad (3.6)$$

The only other operator of bare dimension 2 in the $\mathcal{N} = 4$ theory is an unprotected one, the superconformal primary operator, $K(x)$, belonging to the long Konishi multiplet [16–18]. $K(x)$ is a $SU(4)_R$ singlet and takes the form

$$K = \text{Tr}(\varphi^i\varphi^i) = \frac{1}{4}\varepsilon^{mnpq}\text{Tr}(\bar{\varphi}_{mn}\bar{\varphi}_{pq}) . \quad (3.7)$$

Using (3.4) we can construct a super-operator containing $K(x)$ as $\theta = \bar{\theta} = 0$ component. We define

$$\mathcal{K}(z) = -\frac{1}{8}\varepsilon^{mnpq}\text{Tr}\left[(\bar{d}_p\bar{d}_q\Phi(z))(\bar{d}_r\bar{d}_s\Phi(z))\right] , \quad (3.8)$$

so that

$$K(x) = [\mathcal{K}(z)]_{\theta=\bar{\theta}=0} . \quad (3.9)$$

¹It is in fact an extra-short multiplet, with half as many components as an ordinary 1/2 BPS multiplet.

The conformal symmetry of $\mathcal{N} = 4$ SYM constrains the correlation functions of gauge-invariant primary operators [19]. The space-time dependence of two-point functions of primary operators, $\mathcal{O}_k(x)$, is completely fixed by conformal invariance,

$$\langle \mathcal{O}_h(x) \mathcal{O}_k(y) \rangle = \frac{c_{hk}^{(2)}}{(x-y)^{2\Delta_k}}, \quad (3.10)$$

where Δ_k is the common scaling dimension of \mathcal{O}_k and \mathcal{O}_h . In general Δ_k (as well as the coefficient $c_{kh}^{(2)}$) receives quantum corrections,

$$\Delta_k = \Delta_k^{(0)} + \gamma_k(g), \quad (3.11)$$

where $\gamma_k(g)$ is the anomalous dimension ². The study of the spectrum of anomalous dimensions of composite operators in $\mathcal{N} = 4$ has been a major focus of activity in recent years, in particular in connection with the emergence of integrability properties [20].

Three-point functions of primary operators are also constrained in their form,

$$\langle \mathcal{O}_h(x) \mathcal{O}_k(y) \mathcal{O}_l(z) \rangle = \frac{c_{hkl}^{(3)}}{(x-y)^{\Delta_h+\Delta_k-\Delta_l} (x-z)^{\Delta_h+\Delta_l-\Delta_k} (y-z)^{\Delta_k+\Delta_l-\Delta_h}}, \quad (3.12)$$

where the numerator, $c_{hkl}^{(3)}$, is related to the Operator Product Expansion (OPE) coefficients for the three operators and in general receives quantum corrections.

As mentioned earlier, BPS operators in $\mathcal{N} = 4$ SYM have special non-renormalisation properties. For such operators the BPS condition implies a relation between their $SU(4)_R$ quantum numbers and their scaling dimensions which, as a consequence, do not receive quantum corrections. This fact in turn is related to the absence of quantum corrections to the two-point correlation functions of BPS operators [21]. Similarly three-point functions of BPS operators in $\mathcal{N} = 4$ SYM are tree-level exact and this is related to the absence of quantum corrections to the OPE coefficients among triplets of protected operators [22]. Four- and higher-point functions of protected operators do receive non-trivial quantum corrections both in perturbation theory [23–25] and from instantons [26], but they are ultra-violet finite.

In general all correlation functions of non-protected operators receive quantum corrections, including two- and three-point functions. Even in a finite and exactly conformally invariant theory such as $\mathcal{N} = 4$ SYM, these corrections are accompanied by ultra-violet infinities [18]. These divergences are an artefact of the perturbative expansion and can be reabsorbed into the renormalisation of scaling dimensions and OPE coefficients. However, their presence implies the need to introduce a regularisation scheme even in $\mathcal{N} = 4$ SYM and thus leads to additional subtleties.

In the present paper we consider only correlators of protected operators, focussing on a four-point function of the \mathcal{Q}^{ij} defined in (3.5). In the following section we present the tree-level and one-loop calculations for this four-point function and in deriving our results we will assume the non-renormalisation of two- and three-point functions discussed above. Although the non-renormalisation results were obtained in covariant gauges, the absence of corrections to scaling dimensions and OPE coefficients of BPS operators is a gauge-invariant result and thus remains valid when working in light-cone superspace. Moreover, the fact that these

²This is a slight oversimplification. An accurate definition of the anomalous dimension involves the diagonalisation of the set of two-point functions (3.10), *i.e.* the resolution of the mixing among operators of the same bare dimension.

scalar operators have the same explicit form as in covariant gauges should also ensure the absence of non-physical quantum corrections, such as those associated with a wave function renormalisation, to their two- and three-point functions.

Loop corrections to two- and three-point functions of protected operators involve divergent integrals, which arise in conjunction with vanishing coefficients. Therefore an explicit proof of the non-renormalisation of these correlators requires the introduction of a suitable regularisation. We intend to return to a more detailed analysis of this matter in a future publication, where we will address the issues associated with the regularisation of light-cone superspace calculations in the context of the study of more general non-protected correlation functions.

One of the benefits of superspace formulations of supersymmetric gauge theories is the possibility of providing a compact description of entire multiplets in terms of superfields. In this respect the light-cone superspace description of $\mathcal{N} = 4$ SYM is particularly interesting as it is the only formulation of the theory in which the full $\mathcal{N} = 4$ supersymmetry is manifest. Working with super-operators such as (3.5) and (3.8) should make it possible to extract all correlation functions of operators in the same supersymmetry multiplet from a single super-correlator. It will be interesting to study other components in the θ -expansion of the super-correlation function considered in the next section. These should contain information about correlation functions of the super-partners of the Q^{ij} 's.

The multiplet starting with the superconformal primary operator (3.1)-(3.2) contains the conserved currents associated with the PSU(2,2|4) superconformal symmetry of $\mathcal{N} = 4$ SYM, *i.e.* the energy-momentum tensor, the supersymmetry and R-symmetry currents, as well as other bosonic and fermionic operators for a total of 128+128 components. All these operators are written as SU(N) traces of products of two, three or four elementary fields in the $\mathcal{N} = 4$ multiplet.

Although in this paper we are concerned only with correlation functions of the superconformal primaries (3.1)-(3.2), it is natural to speculate that the light-cone superspace formalism will permit a description of the entire energy-momentum tensor multiplet using a single composite superfield. This will require the addition of terms cubic and quartic in the superfield Φ to the super-operator (3.5). These additional terms should not modify the $\theta = \bar{\theta} = 0$ component, while producing the correct cubic and quartic terms in the remaining operators. The exact form of these additional terms in the super-operator should be determined by the entire $\mathcal{N} = 4$ superalgebra, including the non-linearly realised dynamical generators. The possibility of constructing such a composite superfield operator is intriguing and we hope to investigate it further.

4 A simple four-point correlation function

The study of four-point correlation functions of protected operators in $\mathcal{N} = 4$ SYM provides an ideal testing ground for the application of light-cone superspace techniques to the calculation of off-shell observables. As mentioned in the previous section, four-point functions of BPS operators are less constrained by (super)conformal invariance than two- and three-point functions. They receive quantum corrections, but are free of both infra-red and ultra-violet divergences for generic positions of the operator insertions.

In the case of four-point functions of $\mathcal{N} = 4$ primary operators the dependence on the external points is not fixed by the symmetries of the theory. Quantum corrections to these

correlators can be reorganised into functions, $F_4(r, s; g)$, of the coupling constant and two conformally invariant cross ratios, which can be chosen as

$$r = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad s = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (4.1)$$

where $x_{ij}^2 = (x_i - x_j)^2$. The functions $F_4(r, s; g)$ are unrestricted by the $\text{PSU}(2, 2|4)$ global symmetry group and, in general, receive an infinite series of perturbative and instanton corrections. It is also worth recalling that four-point correlation functions of protected operators contain information on anomalous dimensions and OPE coefficients of non-protected operators in the $\mathcal{N} = 4$ SYM spectrum, which can be extracted from singularities arising in short distance limits, $x_{ij}^2 \rightarrow 0$, for pairs of external points [27].

We consider four-point correlation functions of the operators Q^{ij} given in (3.3),

$$G_4^{(Q)}(x_1, \dots, x_4) = \langle Q^{i_1 j_1}(x_1) Q^{i_2 j_2}(x_2) Q^{i_3 j_3}(x_3) Q^{i_4 j_4}(x_4) \rangle, \quad (4.2)$$

which can be obtained from the correlation functions of the corresponding super-operators, \mathcal{Q}^{ij} , defined as

$$\mathcal{G}_4^{(Q)}(z_1, \dots, z_4) = \langle \mathcal{Q}^{i_1 j_1}(z_1) \mathcal{Q}^{i_2 j_2}(z_2) \mathcal{Q}^{i_3 j_3}(z_3) \mathcal{Q}^{i_4 j_4}(z_4) \rangle, \quad (4.3)$$

by setting to zero the external fermionic coordinates,

$$G_4^{(Q)}(x_1, \dots, x_4) = \mathcal{G}_4^{(Q)}(z_1, \dots, z_4) \Big|_{\theta^{(\alpha)m} = \bar{\theta}_m^{(\alpha)} = 0}, \quad \forall \alpha = 1, \dots, 4, \quad m = 1, \dots, 4, \quad (4.4)$$

where the index α labels the external points.

The super-operators \mathcal{Q}^{ij} defined in (3.5) and their lowest components (3.3) transform in the representation $\mathbf{20}'$ of $\text{SU}(4)_R$. As a consequence there are in principle six independent four-point functions of the type in (4.3), (4.2), since the singlet enters with multiplicity 6 in the tensor product of four $\mathbf{20}'$'s. However, explicit perturbative and instanton calculations indicate that there exist functional relations among any six four-point functions that can be chosen as a basis, leaving only two independent structures. Therefore all correlation functions in (4.2) are determined by two independent functions, $F_4^{(1)}(r, s; g)$ and $F_4^{(2)}(r, s; g)$, of the cross ratios (4.1).

In this paper we restrict our attention to a simple four-point function in the class (4.2), which we denote by $G_4^{(H)}(x_1, \dots, x_4)$. It corresponds to the following choice for the flavour indices

$$G_4^{(H)}(x_1, \dots, x_4) = \langle Q^{12}(x_1) Q^{34}(x_2) Q^{34}(x_3) Q^{12}(x_4) \rangle. \quad (4.5)$$

We re-derive the known tree-level and one-loop contributions to (4.5) working in light-cone superspace. Our starting point is thus

$$\mathcal{G}_4^{(H)}(z_1, \dots, z_4) = \langle \mathcal{Q}^{12}(z_1) \mathcal{Q}^{34}(z_2) \mathcal{Q}^{34}(z_3) \mathcal{Q}^{12}(z_4) \rangle, \quad (4.6)$$

which reduces to (4.5) upon setting to zero the external fermionic coordinates.

The simplifications induced by the choice of $\text{SU}(4)_R$ indices in (4.5) will become apparent in the next subsections where we evaluate this particular four-point function at tree-level and one-loop.

We start by writing (4.5) using the form (3.3) for the Q^{ij} operators,

$$\begin{aligned}
G_4^{(H)}(x_1, \dots, x_4) &= \left\langle \text{Tr} [(\varphi^1 \varphi^2)(x_1)] \text{Tr} [(\varphi^3 \varphi^4)(x_2)] \text{Tr} [(\varphi^3 \varphi^4)(x_3)] \text{Tr} [(\varphi^1 \varphi^2)(x_4)] \right\rangle \\
&= \left(\frac{1}{8} \right)^4 \sigma^{1m_1 n_1} \sigma^{2p_1 q_1} \sigma^{3p_2 q_2} \sigma^{4m_2 n_2} \sigma^{3p_3 q_3} \sigma^{4m_3 n_3} \sigma^{1m_4 n_4} \sigma^{2p_4 q_4} \\
&\times \left\langle \text{Tr} [(\bar{\varphi}_{m_1 n_1} \bar{\varphi}_{p_1 q_1})(x_1)] \text{Tr} [(\bar{\varphi}_{m_2 n_2} \bar{\varphi}_{p_2 q_2})(x_2)] \text{Tr} [(\bar{\varphi}_{m_3 n_3} \bar{\varphi}_{p_3 q_3})(x_3)] \text{Tr} [(\bar{\varphi}_{m_4 n_4} \bar{\varphi}_{p_4 q_4})(x_4)] \right\rangle.
\end{aligned} \tag{4.7}$$

The explicit form of the super-operator containing (4.7) as its $\theta = \bar{\theta} = 0$ component is

$$\begin{aligned}
\mathcal{G}_4^{(H)}(z_1, \dots, z_4) &= \frac{1}{16} \left(\frac{1}{8} \right)^4 \left(\frac{i}{\sqrt{2}} \right)^8 \sigma^{1m_1 n_1} \sigma^{2p_1 q_1} \sigma^{3p_2 q_2} \sigma^{4m_2 n_2} \sigma^{3p_3 q_3} \sigma^{4m_3 n_3} \sigma^{1m_4 n_4} \sigma^{2p_4 q_4} \\
&\times \left\langle \left(\bar{d}_{m_1}^{(1)} \bar{d}_{n_1}^{(1)} \Phi^a(z_1) \bar{d}_{p_1}^{(1)} \bar{d}_{q_1}^{(1)} \Phi^a(z_1) \right) \left(\bar{d}_{m_2}^{(2)} \bar{d}_{n_2}^{(2)} \Phi^b(z_2) \bar{d}_{p_2}^{(2)} \bar{d}_{q_2}^{(2)} \Phi^b(z_2) \right) \right. \\
&\times \left. \left(\bar{d}_{m_3}^{(3)} \bar{d}_{n_3}^{(3)} \Phi^c(z_3) \bar{d}_{p_3}^{(3)} \bar{d}_{q_3}^{(3)} \Phi^c(z_3) \right) \left(\bar{d}_{m_4}^{(4)} \bar{d}_{n_4}^{(4)} \Phi^d(z_4) \bar{d}_{p_4}^{(4)} \bar{d}_{q_4}^{(4)} \Phi^d(z_4) \right) \right\rangle.
\end{aligned} \tag{4.8}$$

Notice that in $G_4^{(H)}(x_1, \dots, x_4)$ we choose all the Q^{ij} operators with distinct flavour indices, so that when re-writing them in the form (3.3) the second term, which subtracts the $\text{SU}(4)_R$ trace never appears. This leads to simplifications in the calculation since there are fewer contractions to consider. In addition, divergences in intermediate steps are avoided. The flavour trace that is subtracted in (3.3) is in fact proportional to the unprotected Konishi operator (3.7), whose four-point functions are divergent at any fixed order in perturbation theory. These divergences would be needed, had we chosen to insert Q^{ij} operators with $i = j$, to cancel other divergent contributions and ensure the finiteness of the resulting $G_4^{(Q)}$ four-point function.

For compactness of notation, in the following we write the super-propagator as

$$\Delta_j^i(z_1 - z_2) = \frac{k \delta_j^i}{x_{12}^2} \langle d^4 \rangle \delta_{12}^{(8)}, \tag{4.9}$$

where $k = -2/(2\pi)^2 (4!)^3$, $x_{12}^2 = (x_1 - x_2)^2$ and $\delta_{12}^{(8)} = \delta^{(4)}(\theta_1 - \theta_2) \delta^{(4)}(\bar{\theta}_1 - \bar{\theta}_2)$.

4.1 Tree level

At tree level there are multiple contractions possible in (4.8). However, only the one shown in figure 1 is non-zero. The reason why all other contractions vanish is evident from the form of $G_4^{(H)}$ in the first line of (4.7): all other contraction are zero because the propagator (B.29) for the elementary scalars is diagonal in flavour space.

It is straightforward to obtain the same result in superspace. A free propagator connecting scalars $\varphi^{a_1 i_1}(x_1)$ and $\varphi^{a_2 i_2}(x_2)$ in two Q operators gives rise to the factor

$$\sigma^{i_1 m_1 n_1} \sigma^{i_2 m_2 n_2} \left(\bar{d}_{m_1} \bar{d}_{n_1} \langle d^4 \rangle \frac{\delta_{12}^{(8)}}{x_{12}^2} \overleftarrow{\bar{d}_{m_2} \bar{d}_{n_2}} \delta^{a_1 a_2} \right), \tag{4.10}$$

which, upon setting to zero the θ and $\bar{\theta}$ coordinates at points z_1 and z_2 , reduces to

$$\frac{\delta^{a_1 a_2}}{x_{12}^2} \sigma^{i_1 m_1 n_1} \sigma^{i_2 m_2 n_2} \langle d^4 \rangle \bar{d}_{m_1} \bar{d}_{n_1} \bar{d}_{m_2} \bar{d}_{n_2} \delta_{12}^{(8)} \Big|_{\theta_1 = \bar{\theta}_1 = 0} = (4!)^3 8 \frac{\delta^{a_1 a_2}}{x_{12}^2} \delta^{i_1 i_2}, \tag{4.11}$$

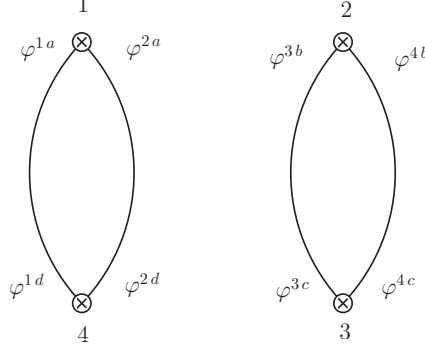


Figure 1: Tree-level contribution to $G_4^{(H)}(x_1, \dots, x_4)$.

where $\sigma^{i_1 m_1 n_1} \sigma^{i_2 m_2 n_2} \varepsilon_{m_1 n_1 m_2 n_2} = 8 \delta^{i_1 i_2}$. Thus each external φ^i can only be connected through a free propagator to a φ^j with $i = j$ for a non-vanishing contribution. Therefore at tree level the only allowed contraction in $\mathcal{G}_4^{(H)}(z_1, \dots, z_4)$ is the one in figure 1 which, using (4.9), yields

$$\begin{aligned} \mathcal{G}_4^{(H)}(z_1, \dots, z_4) &= \frac{1}{2^{20}} \sigma^{1m_1 n_1} \sigma^{2p_1 q_1} \sigma^{3p_2 q_2} \sigma^{4m_2 n_2} \sigma^{3p_3 q_3} \sigma^{4m_3 n_3} \sigma^{1m_4 n_4} \sigma^{2p_4 q_4} \\ &\times k^4 \left(\bar{d}_{m_1} \bar{d}_{n_1} \langle d^4 \rangle \frac{\delta_{14}^{(8)}}{x_{14}^2} \overleftarrow{\bar{d}_{n_4} \bar{d}_{m_4}} \delta^{ad} \right) \left(\bar{d}_{p_1} \bar{d}_{q_1} \langle d^4 \rangle \frac{\delta_{14}^{(8)}}{x_{14}^2} \overleftarrow{\bar{d}_{q_4} \bar{d}_{p_4}} \delta^{ad} \right) \\ &\times \left(\bar{d}_{m_2} \bar{d}_{n_2} \langle d^4 \rangle \frac{\delta_{23}^{(8)}}{x_{23}^2} \overleftarrow{\bar{d}_{n_3} \bar{d}_{m_3}} \delta^{bc} \right) \left(\bar{d}_{p_2} \bar{d}_{q_2} \langle d^4 \rangle \frac{\delta_{23}^{(8)}}{x_{23}^2} \overleftarrow{\bar{d}_{q_3} \bar{d}_{p_3}} \delta^{bc} \right). \end{aligned} \quad (4.12)$$

Setting to zero all the external θ^m 's and $\bar{\theta}_m$'s we get

$$\begin{aligned} G_4^{(H)}(x_1, \dots, x_4) &= (\sigma^{1m_1 n_1} \sigma^{1m_4 n_4} \varepsilon_{m_1 n_1 m_4 n_4}) (\sigma^{2p_1 q_1} \sigma^{2p_4 q_4} \varepsilon_{p_1 q_1 p_4 q_4}) (\sigma^{3p_2 q_2} \sigma^{3p_3 q_3} \varepsilon_{p_2 q_2 p_3 q_3}) \\ &\times (\sigma^{4m_2 n_2} \sigma^{4m_3 n_3} \varepsilon_{m_2 n_2 m_3 n_3}) \frac{1}{2^{20}} k^4 (N^2 - 1)^2 (4!)^{12} \frac{1}{(x_{14}^2)^2 (x_{23}^2)^2}, \end{aligned} \quad (4.13)$$

where we used $\delta^{aa} = N^2 - 1$. Simplifying (4.13) and substituting $k = -2/(2\pi)^2 (4!)^3$ we get

$$\left[G_4^{(H)}(x_1, \dots, x_4) \right]_{\text{tree}} = \frac{(N^2 - 1)^2}{16(2\pi)^8} \frac{1}{(x_{14}^2)^2 (x_{23}^2)^2}. \quad (4.14)$$

4.2 One-loop

One-loop contributions to $G_4^{(H)}(x_1, \dots, x_4)$ are of order g^2 and involve either two cubic interaction vertices or a single quartic vertex. Moreover we can distinguish between disconnected diagrams, which factorise into the product of tree-level and one-loop two-point functions, and connected four-point diagrams.

4.2.1 Factorised two-point functions

Figure 2 depicts the disconnected one-loop contributions to $G_4^{(H)}$. They factorise as

$$\langle \mathcal{Q}^{12}(z_1) \mathcal{Q}^{12}(z_4) \rangle_{1\text{-loop}} \langle \mathcal{Q}^{34}(z_2) \mathcal{Q}^{34}(z_3) \rangle_{\text{tree}}. \quad (4.15)$$

A second set of diagrams in which the interaction vertices connect to the external points z_2 and z_3 gives rise to a contribution of the form

$$\langle \mathcal{Q}^{12}(z_1) \mathcal{Q}^{12}(z_4) \rangle_{\text{tree}} \langle \mathcal{Q}^{34}(z_2) \mathcal{Q}^{34}(z_3) \rangle_{1\text{-loop}}. \quad (4.16)$$

Both (4.15) and (4.16) vanish thanks to the non-renormalisation of two-point functions of protected operators. Therefore we assume that $G_4^{(H)}(x_1, \dots, x_4)$ receives no contribution from the sum of all diagrams with the topologies in figure 2. While this assumption is justified because the vanishing of one-loop corrections to two point functions of BPS operators is a gauge-independent result, it would be desirable to have an explicit proof in light-cone superspace and we intend to revisit this issue.

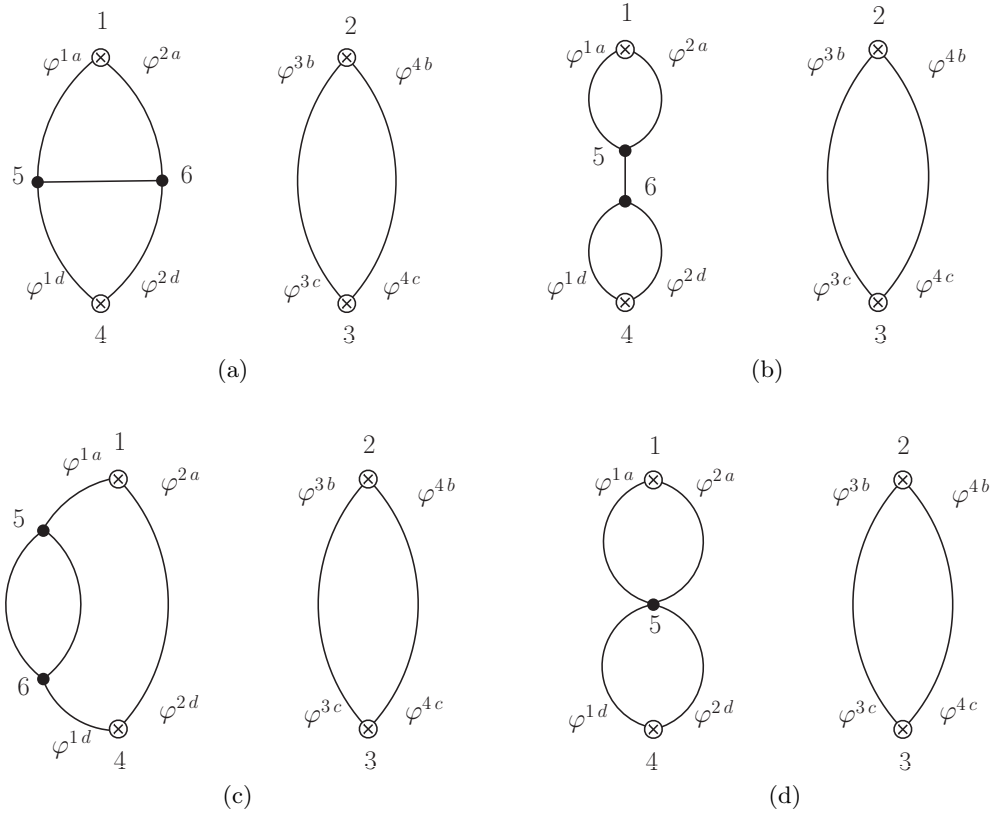


Figure 2: Disconnected one-loop contributions to $G_4^{(H)}(x_1, \dots, x_4)$.

4.2.2 Connected diagrams involving two cubic vertices

The next set of diagrams of order g^2 that we need to consider are connected ones involving two cubic vertices. There are two distinct types of contractions to take into account which are shown in figure 3.

The building blocks for these diagrams are the cubic vertices (2.17) and (2.18). Analysing the combinations of chiral derivatives in these vertices one can verify that in order to produce a potentially non-vanishing contribution a diagram must involve one vertex of each type. This is proven in appendix D.1, where we also discuss an explicit example. A consequence of this observation is the reality of individual contributions to the four-point functions $G_4^{(Q)}$, as the two cubic vertices (2.17) and (2.18) are complex conjugates of each other.

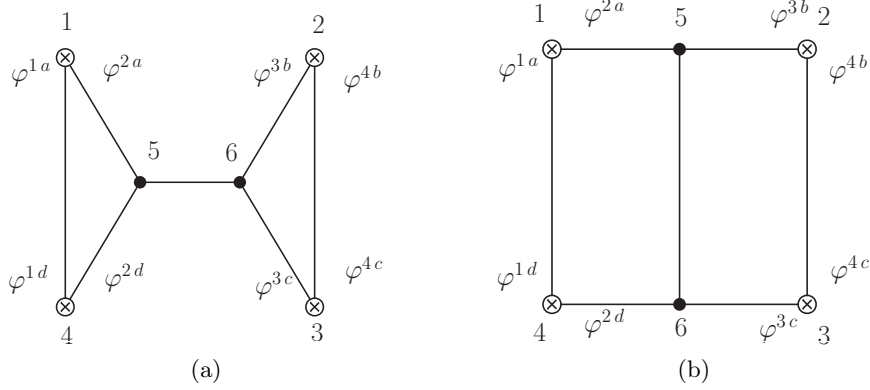


Figure 3: Connected one-loop contributions to $G_4^{(H)}(x_1, \dots, x_4)$ involving cubic vertices.

The contributions from the two diagrams in figure 3 vanish individually, but for different reasons.

The vanishing of diagrams of the type in figure 3a is straightforward. Since the superfield propagator is diagonal in colour space, the free contractions between points z_1 and z_4 and between points z_2 and z_3 , combined with the traces at each external point, force two of the indices of the totally antisymmetric structure constants f^{abc} at the interaction vertices in z_5 and z_6 to be the same. Therefore these diagrams are identically zero. Since this vanishing result follows from the colour structure of the diagram, all other Wick contractions, which differ only in the distribution of flavour indices, give a zero result as well.

Diagrams of the type shown in figure 3b also vanish, but the proof is slightly more involved, requiring manipulations which are described in detail in appendix D.1. The vanishing of contributions with this topology follows from the observation that a contraction in which two external fields φ^{i_1} and φ^{i_2} are connected to a cubic interaction vertex gives rise to a factor of $\sigma^{i_1 mn} \sigma^{i_2 pq} \varepsilon_{mnpq} = 8\delta^{i_1 i_2}$. The reason for this is explained under Rule D.1 in Appendix D.1.

In the case of the diagram in figure 3b the internal point z_5 (z_6) connects φ^2 with φ^3 , which results in a factor of $\sigma^{2mn} \sigma^{3pq} \varepsilon_{mnpq} = 0$. Other Wick contractions, with a different distribution of flavour indices, vanish for the same reason.

In general since $\sigma^{i_1 mn} \sigma^{i_2 pq} \varepsilon_{mnpq} = 0$ for $i_1 \neq i_2$, scalar fields φ^{i_1} and φ^{i_2} with $i_1 \neq i_2$ cannot be connected through a cubic vertex.

4.2.3 Connected diagrams involving one quartic vertex

The last type of contribution to $G_4^{(H)}(x_1, \dots, x_4)$ at order g^2 comes from diagrams involving a single quartic vertex. With our choice of external flavours the only allowed topology is depicted in figure 4, where the interaction vertex at point z_5 can be either (2.19) or (2.20).

The first type of contribution, constructed using the vertex (2.19), vanishes. Therefore the entire one-loop correction to $G_4^{(H)}(x_1, \dots, x_4)$ comes from diagrams of the type in figure 4, with the quartic interaction at point z_5 corresponding to Vertex 4-II (2.20).

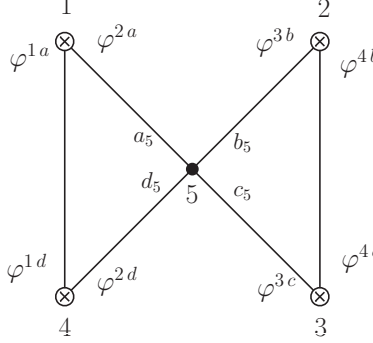


Figure 4: Connected one-loop contributions to $G_4^{(H)}(x_1, \dots, x_4)$ involving a quartic vertex (a_5 , b_5 , c_5 and d_5 are colour indices).

We present in detail the calculation of the contraction shown in the figure, in which the two free propagators connecting points z_1 and z_4 and points z_2 and z_3 carry flavour 1 and 4 respectively. There are additional contributions in which the $z_1 - z_4$ line has flavour 2 and/or the $z_2 - z_3$ line has flavour 3. These produce the same contribution as the diagram we analyse and therefore simply give rise to a multiplicity factor in the final answer.

The vanishing of diagrams involving Vertex 4-I (2.19) can be understood as follows. In this figure, colour labels a_5 and b_5 (c_5 and d_5) cannot sit on both the interaction legs to the left, or both the legs to the right, else the structure function at the interaction point, $f^{ea_5b_5} f^{ec_5d_5}$ will vanish. This is due to the external propagator connecting x_1 to x_4 (x_2 to x_3) – the Kronecker delta in the propagator forces the two colour indices on the two left (right) legs of the interaction vertex to be the same.

However, unless a_5 and b_5 (c_5 and d_5) sit on both the left legs or both the right legs, we will run into a contraction of the form $\sigma^{2mn} \sigma^{3pq} \varepsilon_{mnpq}$, which is zero. The reason why we end up with this contraction is explained under Rule D.2 in Appendix D.2. Thus the requirement that the structure functions be non-zero conflicts with the requirement that the $\sigma\sigma\varepsilon$ contractions be non-zero. Consequently Vertex 4-I does not contribute.

Finally we come to the calculation of the non-zero contribution from diagrams of the topology in figure 4 in which the interaction vertex is of type 4-II.

We factorise the diagram as in figure 5. The different Wick contractions correspond to inequivalent ways of gluing together parts (a) and (b) in the figure.

The following contribution comes from figure 5a and is common to all diagrams in this set

$$E_4[a_5, b_5, c_5, d_5] = \frac{1}{16} \left(\frac{1}{8} \right)^4 \sigma^{1m_1n_1} \sigma^{2p_1q_1} \sigma^{3p_2q_2} \sigma^{4m_2n_2} \sigma^{3p_3q_3} \sigma^{4m_3n_3} \sigma^{1m_4n_4} \sigma^{2p_4q_4} k^6 \delta^{ad} \delta^{bc} \\ \times \left(\frac{i}{\sqrt{2}} \right)^8 \left(\bar{d}_{m_1} \bar{d}_{n_1} \langle d^4 \rangle \frac{\delta_{14}^8}{x_{14}^2} \overleftarrow{\bar{d}_{n_4} \bar{d}_{m_4}} \right) \left(\bar{d}_{m_2} \bar{d}_{n_2} \langle d^4 \rangle \frac{\delta_{23}^8}{x_{23}^2} \overleftarrow{\bar{d}_{n_3} \bar{d}_{m_3}} \right) f^{ea_5b_5} f^{ec_5d_5} \left(-\frac{g^2}{64} \right). \quad (4.17)$$

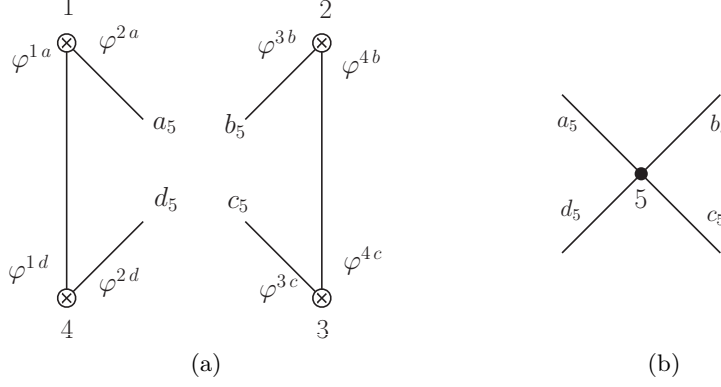


Figure 5: Factorisation of diagram involving a quartic vertex.

This common portion simplifies to

$$E_4[a_5, b_5, c_5, d_5] = T(\sigma) \left(-\frac{g^2}{2^{26}} \right) k^6 \delta^{ad} \delta^{bc} f^{ea_5 b_5} f^{ec_5 d_5} \frac{(4!)^6}{8^4} \varepsilon_{m_1 n_1 n_4 m_4} \varepsilon_{m_2 n_2 n_3 m_3} \frac{1}{x_{14}^2} \frac{1}{x_{23}^2}, \quad (4.18)$$

where $T(\sigma)$ denotes the product of the eight σ coefficients in (4.17).

We now need to consider all possible ways of gluing of this factor with the piece resulting from figure 5b. We use the following notation,

$$\begin{array}{c} a_5 \quad b_5 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ d_5 \quad c_5 \end{array} \quad \equiv V_4[a_5, b_5, c_5, d_5],$$

where the order of the arguments in V_4 corresponds to the clockwise labelling in the vertex starting from the top left leg.

The different Wick contractions are analysed in appendix D.2. Combining all the non-zero contributions we find that figure 4 evaluates to

$$-g^2 f^{abc} f^{abc} \frac{1}{8(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2} \int d^4 x_5 \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}. \quad (4.19)$$

Using $f^{abc} f^{abc} = N(N^2 - 1)$ and including all multiplicity factors the complete one-loop contribution to (4.5) is therefore

$$\left[G_4^{(H)}(x_1, \dots, x_4) \right]_{1\text{-loop}} = -g^2 N(N^2 - 1) \frac{1}{2(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2} \int d^4 x_5 \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}. \quad (4.20)$$

The box integral in (4.20) is well known [28] and can be expressed in terms of the cross ratios (4.1). Using the form of the box integral in [23], the one-loop contribution to $G_4^{(H)}(x_1, \dots, x_4)$ takes the form

$$\left[G_4^{(H)}(x_1, \dots, x_4) \right]_{1\text{-loop}} = -g^2 N(N^2 - 1) \pi^2 \frac{1}{2(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2 x_{13}^2 x_{24}^2} F_4^{(H)}(r, s), \quad (4.21)$$

where $F_4^{(H)}(r, s)$ can be expressed as a combination of logarithms and dilogarithms as

$$F_4^{(H)}(r, s) = \frac{1}{\sqrt{p}} \left\{ \log(r) \log(s) - \left[\log \left(\frac{r + s - 1 - \sqrt{p}}{2} \right) \right]^2 + \right. \\ \left. - 2 \operatorname{Li}_2 \left(\frac{2}{1 + r - s + \sqrt{p}} \right) - 2 \operatorname{Li}_2 \left(\frac{2}{1 - r + s + \sqrt{p}} \right) \right\}, \quad (4.22)$$

where $\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ and

$$p = 1 + r^2 + s^2 - 2r - 2s - 2rs. \quad (4.23)$$

5 Open problems and future directions

In this paper we have initiated a program aimed at systematically studying correlation functions of gauge-invariant operators in $\mathcal{N} = 4$ SYM using the light-cone superspace formulation. Our main goals are on the one hand to develop efficient techniques for the computation of perturbative corrections to correlation functions and on the other to shed light on subtleties which can potentially arise from the use of the light-cone gauge in the calculation of off-shell quantities.

As a computational tool light-cone superspace is particularly promising for a number of reasons. This formulation of the $\mathcal{N} = 4$ SYM theory uses only one type of superfield, which carries no space-time or $\mathrm{SU}(4)_R$ indices. Therefore the general structure of super Feynman diagrams and the combinatorial analysis involved in their study are simpler than in other formulations. Moreover, while the one-loop calculation we presented did not show noticeable simplifications compared to similar covariant calculations, we expect that the manifest $\mathcal{N} = 4$ supersymmetry will lead to a significant computational advantage, in terms of the number of diagrams to evaluate, at higher orders in the perturbative expansion.

Particularly interesting is the possibility of taking advantage of the full $\mathcal{N} = 4$ supersymmetry to describe in compact form entire multiplets of operators and their correlation functions. This should be possible for the multiplet of the energy-momentum tensor, which is expected to take the form of a linear combination of quadratic, cubic and quartic terms in the $\mathcal{N} = 4$ superfield, Φ . In order to determine the exact combination as well as to generalise such a construction to different multiplets, it will be important to better understand the role played by the non-linearly realised dynamical supersymmetries.

In the case of the simple four-point function $G_4^{(H)}(x_1, \dots, x_4)$ we reproduced the known result to one-loop order. The light-cone gauge thus yields a manifestly Lorentz covariant result. This is thanks to non-trivial cancellations of derivatives and $1/\partial_-$ factors. It will be important to understand these cancellations in a systematic way for more complicated correlation functions and/or at higher orders in perturbation theory.

A distinct, but related, issue concerns the general consistency of the light-cone gauge formalism, in its superspace realisation, when applied to the study of off-shell observables. In the case of a (super) conformal gauge theory such as $\mathcal{N} = 4$ SYM the potential subtleties are associated with spurious infra-red divergences induced by the presence of the $1/\partial_-$ operators. In the case of the simple four-point function that we studied in this paper, various cancellations ensured the absence of any such singularities from the final result. At this stage we do not yet have a clear understanding of how (or even if) similar cancellations take place

in general perturbative calculations. The question of whether or not spurious infra-red divergences arise in generic gauge-invariant correlation function is therefore still open and this is an aspect that deserves further consideration.

Another important point that remains to be addressed is the identification of the most convenient regularisation method to deal with divergent integrals. Previous applications of light-cone superspace, including the all order proof the ultra-violet finiteness of $\mathcal{N} = 4$ SYM, did not require the use of an explicit regularisation. However, divergences do arise in the calculation of correlation functions of non-protected operators such as the Konishi operator (3.7)-(3.8). Hence a suitable regularisation scheme will be needed for such calculations. As an added benefit this will also make it possible to explicitly prove the non-renormalisation of two- and three-point functions of protected operators.

We consider the results presented in this paper to be encouraging and we hope to address the open questions outlined above in future publications.

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A Conventions and notation

We work with space-time signature $(-, +, +, +)$ and define the light-cone coordinates and their derivatives as

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad x = \frac{1}{\sqrt{2}}(x^1 + i x^2), \quad \bar{x} = \frac{1}{\sqrt{2}}(x^1 - i x^2), \quad (\text{A.1})$$

$$\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3), \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 - i \partial_2), \quad \partial = \frac{1}{\sqrt{2}}(\partial_1 + i \partial_2). \quad (\text{A.2})$$

The gauge field components in the light-cone decomposition are

$$A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_3), \quad A = \frac{1}{\sqrt{2}}(A_1 + i A_2), \quad \bar{A} = \frac{1}{\sqrt{2}}(A_1 - i A_2). \quad (\text{A.3})$$

The light-cone gauge fixing involves setting $A_- = 0$ and integrating out A_+ , leaving only the two transverse physical components, A and \bar{A} . The four Weyl fermions in the $\mathcal{N} = 4$ multiplet, ψ_α^m , and their conjugates, $\bar{\psi}_{m\dot{\alpha}}$, are decomposed according to the projection

$$\psi_\alpha^m \rightarrow \psi_{(\pm)}^m = \mathcal{P}_\pm \psi_\alpha^m, \quad \bar{\psi}_{m\dot{\alpha}} \rightarrow \bar{\psi}_m^{(\pm)} = \mathcal{P}_\pm \bar{\psi}_{m\dot{\alpha}}, \quad (\text{A.4})$$

where $\mathcal{P}_\pm = -\frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^3)$. The $\psi_{(+)}^m$ and $\bar{\psi}_m^{(+)}$ components can be integrated out. The light-cone description uses the remaining one-component fermionic fields $\lambda^m \equiv \psi_{(-)}^m$, and their conjugates, $\bar{\lambda}_m \equiv \bar{\psi}_m^{(-)}$.

The Grassmann integrals in light-cone superspace are normalised so that

$$\int d\theta_m \theta^n = \delta_m^n, \quad \int d\bar{\theta}^m \bar{\theta}_n = \delta_n^m. \quad (\text{A.5})$$

We define

$$d^4\theta = \frac{1}{(4!)^2} \varepsilon^{mnpq} d\theta_m d\theta_n d\theta_p d\theta_q, \quad d^4\bar{\theta} = \frac{1}{(4!)^2} \varepsilon_{mnpq} d\bar{\theta}^m d\bar{\theta}^n d\bar{\theta}^p d\bar{\theta}^q. \quad (\text{A.6})$$

This, together with (A.5), ensures that

$$\int d^4\theta \delta^{(4)}(\theta) = \int d^4\bar{\theta} \delta^{(4)}(\bar{\theta}) = 1, \quad (\text{A.7})$$

where the δ -functions are defined as

$$\delta^{(4)}(\theta) = \langle \theta^4 \rangle \equiv \varepsilon_{mnpq} \theta^m \theta^n \theta^p \theta^q, \quad \delta^{(4)}(\bar{\theta}) = \langle \bar{\theta}^4 \rangle \equiv \varepsilon^{mnpq} \bar{\theta}_m \bar{\theta}_n \bar{\theta}_p \bar{\theta}_q. \quad (\text{A.8})$$

The scalar fields in the $\mathcal{N} = 4$ multiplet can be represented either as $\text{SU}(4)_R$ bi-spinors, φ^{mn} , satisfying the reality condition (2.1) or as vectors, φ^i , $i = 1, \dots, 6$. The two representations are related by

$$\varphi^i = \frac{1}{\sqrt{8}} \Sigma_{mn}^i \varphi^{mn} = \frac{1}{2\sqrt{8}} \varepsilon^{mnpq} \Sigma_{mn}^i \bar{\varphi}_{pq} = \frac{1}{\sqrt{8}} \sigma^{ipq} \bar{\varphi}_{pq}. \quad (\text{A.9})$$

where Σ_{mn}^i ($\bar{\Sigma}_i^{mn}$) are Clebsch-Gordan coefficients relating the product of two $\mathbf{4}$'s ($\bar{\mathbf{4}}$'s) to the $\mathbf{6}$ of $\text{SU}(4)$. They are defined as follows

$$\begin{aligned} \Sigma_{mn}^I &= (\Sigma_{mn}^I, \Sigma_{mn}^{I+3}) = (\eta_{mn}^I, i\bar{\eta}_{mn}^I), \\ \bar{\Sigma}_i^{mn} &= (\bar{\Sigma}_i^{mn}, \bar{\Sigma}_{i+3}^{mn}) = (\eta_{mn}^I, -i\bar{\eta}_{mn}^I), \quad I = 1, 2, 3, \end{aligned} \quad (\text{A.10})$$

where $(\eta_{mn}^I, i\bar{\eta}_{mn}^I)$ are 't Hooft symbols,

$$\begin{aligned} \eta_{mn}^I &= \bar{\eta}_{mn}^I = \varepsilon_{Imn}, \quad m, n = 1, 2, 3 \\ \eta_{m4}^I &= \bar{\eta}_{4m}^I = \delta_m^I, \quad m = 1, 2, 3 \\ \eta_{mn}^I &= -\eta_{nm}^I, \quad \bar{\eta}_{mn}^I = -\bar{\eta}_{nm}^I. \end{aligned} \quad (\text{A.11})$$

Splitting up the i index in terms of $I = 1, 2, 3$, the coefficients (A.10) can be written as

$$\begin{aligned} \Sigma_{mn}^I &= \varepsilon_{mn4}^I + (\delta_m^I \delta_n^4 - \delta_n^I \delta_m^4), \\ \Sigma_{mn}^{I+3} &= i\varepsilon_{mn4}^I - i(\delta_m^I \delta_n^4 - \delta_n^I \delta_m^4). \end{aligned} \quad (\text{A.12})$$

From this we obtain the σ^{imn} coefficients

$$\begin{aligned} \sigma^{Ipq} &= \varepsilon^{Ipq4} + (\delta^{Ip} \delta_4^q - \delta_4^p \delta^{Iq}), \\ \sigma^{(I+3)pq} &= -i\varepsilon^{Ipq4} + i(\delta^{Ip} \delta_4^q - \delta_4^p \delta^{Iq}). \end{aligned} \quad (\text{A.13})$$

B Derivation of super-propagator

In this appendix we discuss in detail the derivation of the propagator (2.16) for the $\mathcal{N} = 4$ superfield. We start with a path integral derivation which will allow us to check the consistency of various conventions for Grassmann integrals and functional derivatives.

B.1 Path integral derivation

The superfield propagator can be obtained inverting the kinetic operator in (2.10). We can obtain it constructing the generating functional for Green functions of the $\mathcal{N} = 4$ superfield in the free theory limit, $Z_0[J]$.

Functional differentiation of $Z[J]$ with respect to the sources, $J(x, \theta, \bar{\theta})$, gives rise to Green functions of the $\mathcal{N} = 4$ superfields. Because of the chirality of both Φ and J we need to be careful in defining the rules for functional differentiation in superspace. In defining the functional derivative with respect to a chiral superfield we require the condition that the variation of a chiral superfield be chiral. To satisfy this condition we consider a chiral superfield, $\Psi(x, \theta, \bar{\theta})$, written in terms of the chiral variable (2.6) and we impose

$$\frac{\delta \Psi(y', \theta')}{\delta \Psi(y, \theta)} = \delta^{(4)}(y - y') \delta^{(4)}(\theta - \theta'). \quad (\text{B.1})$$

To obtain the form of the derivative $\delta \Psi(x', \theta', \bar{\theta}') / \delta \Psi(x, \theta, \bar{\theta})$ in terms of the standard superspace coordinates we consider

$$\frac{\delta}{\delta \Psi(x, \theta, \bar{\theta})} \int d^4 x' d^4 \theta' d^4 \bar{\theta}' \Psi(x', \theta', \bar{\theta}') F(x', \theta', \bar{\theta}'), \quad (\text{B.2})$$

where $F(x, \theta, \bar{\theta})$ is a generic (non-chiral) superfield. Using (B.1) we can evaluate (B.2) as follows

$$\begin{aligned} & \frac{\delta}{\delta \Psi(x, \theta, \bar{\theta})} \int d^4 x' d^4 \theta' d^4 \bar{\theta}' \Psi(x', \theta', \bar{\theta}') F(x', \theta', \bar{\theta}') \\ &= \int d^4 y' d^4 \theta' d^4 \bar{\theta}' \frac{\delta \Psi(y', \theta')}{\delta \Psi(y, \theta)} F(x'^+, y'^-, \frac{i}{\sqrt{2}} \theta' \bar{\theta}', x', \bar{x}', \theta', \bar{\theta}') \\ &= \int d^4 \bar{\theta}' F(x^+, y^- + \frac{i}{\sqrt{2}} \theta \bar{\theta}', x, \bar{x}, \theta, \bar{\theta}') \\ &= \frac{1}{(4!)^2} \langle d^4 \rangle F(x, \theta, \bar{\theta}), \end{aligned} \quad (\text{B.3})$$

where in the last step we used

$$\int d\bar{\theta}^k F(x^+, y^- + \frac{i}{\sqrt{2}} \theta \bar{\theta}', x, \bar{x}, \theta, \bar{\theta}') = d^k F(x^+, x^-, x, \bar{x}, \theta, \bar{\theta}'), \quad k = 1, \dots, 4, \quad (\text{B.4})$$

which can be verified expanding left and right hand sides in components. From (B.3) we deduce the rule for functional differentiation with respect to a chiral superfield,

$$\frac{\delta \Psi^a(x', \theta', \bar{\theta}')}{\delta \Psi^b(x, \theta, \bar{\theta})} = \frac{1}{(4!)^2} \delta_b^a \langle d^4 \rangle \delta^{(4)}(x - x') \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}'), \quad (\text{B.5})$$

which applies in particular to the $\mathcal{N} = 4$ superfield, Φ . For its conjugate, $\bar{\Phi}$, using (2.9), we get

$$\frac{\delta \bar{\Phi}^a(x', \theta', \bar{\theta}')}{\delta \Phi^b(x, \theta, \bar{\theta})} = \frac{1}{2(4!)^3} \delta_b^a \frac{\langle \bar{d}^4 \rangle \langle d^4 \rangle}{\partial_-^2} \delta^{(4)}(x - x') \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}'). \quad (\text{B.6})$$

We can now define the generating functional, $Z[J]$, as follows

$$Z[J] = \frac{\int [d\Phi] e^{-S[\Phi] + \int d^{12}z \Phi^a(z) \frac{\langle \bar{d}^4 \rangle}{4\partial_-^2} J_a(z)}}{\int [d\Phi] e^{-S[\Phi]}}, \quad (\text{B.7})$$

where, as usual, $d^{12}z = d^4x d^4\theta d^4\bar{\theta}$.

Notice, in particular, the coupling to the sources, $J(z)$, in (B.7). This is chosen so as to produce the correct coupling to external sources in the equations of motion. This can be seen considering the free theory in the presence of external sources,

$$\int d^{12}z \frac{1}{2} \Phi^a(z) \mathcal{K}_a^b \Phi_b(z) + \int d^{12}z \Phi^a(z) \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a(z), \quad (\text{B.8})$$

where the kinetic operator is

$$\mathcal{K}_a^b = -3 \delta_a^b \frac{\langle \bar{d}^4 \rangle \square}{\partial_-^4}. \quad (\text{B.9})$$

Varying (B.8) with respect to the superfield Φ gives rise to the correct equations of motion in the presence of an external source,

$$\frac{1}{(4!)^2} \langle d^4 \rangle \mathcal{K}_a^b \Phi_b(x, \theta, \bar{\theta}) = J_a(x, \theta, \bar{\theta}). \quad (\text{B.10})$$

The right hand side is straightforward to obtain using the definition (B.5),

$$\begin{aligned} \frac{\delta}{\delta \Phi^a(z)} \int d^{12}z' \Phi^b(z') \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_b(z') &= \frac{1}{(4!)^2} \int d^{12}z' \langle d^4 \rangle \delta^{(12)}(z - z') \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a(z') \\ &= \frac{1}{(4!)^2} \int d^{12}z' \delta^{(12)}(z - z') \frac{\langle d^4 \rangle \langle \bar{d}^4 \rangle}{4\partial_-^4} J_a(z') = \int d^{12}z' \delta^{(12)}(z - z') J_a(z') = J_a(z), \end{aligned} \quad (\text{B.11})$$

where we used the fact that $\langle d^4 \rangle \langle \bar{d}^4 \rangle = 4(4!)^2 \partial_-^4$ when acting on a chiral superfield such as $J(z)$.

In the free theory limit the exponent in the generating functional (B.7) reduces to

$$-\frac{1}{2} \left(\Phi^a, \mathcal{K}_a^b \Phi_b \right) + \left(\Phi^a, \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a \right) = -\frac{1}{2} \int d^{12}z \Phi^a(z) \mathcal{K}_a^b \Phi_b(z) + \int d^{12}z \Phi^a(z) \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J_a(z). \quad (\text{B.12})$$

The functional integral (B.7) becomes Gaussian and thus straightforward to compute. The result is

$$Z_0[J] = e^{\frac{1}{2} (\tilde{J}^a, [\mathcal{K}^{-1}]_a^b \tilde{J}_b)} \quad (\text{B.13})$$

where

$$\tilde{J}^a(z) = \frac{\langle \bar{d}^4 \rangle}{4\partial_-^4} J^a(z) \quad (\text{B.14})$$

and \mathcal{K}^{-1} is the inverse of the kinetic operator (B.9). In (B.13) a factor of $\det(\mathcal{K})^{-1/2}$ has been cancelled between numerator and denominator. The free generating functional (B.13) allows to construct the perturbative expansion of the full functional $Z[J]$ in (2.11).

Introducing the kernel, $\Delta(z, z')$, of the operator \mathcal{K}^{-1} , we can rewrite (B.13) as

$$Z_0[J] = e^{\frac{1}{2} \int d^{12}z d^{12}z' \tilde{J}^a(z) [\Delta(z, z')]_a^b \tilde{J}_b(z')}. \quad (\text{B.15})$$

$\Delta(z, z')$ is of course the super-propagator we are interested in. Let us denote by $K(z, z')$ the kernel of the kinetic operator (B.9),

$$K(z, z') = -3 \delta^{(12)}(z - z') \frac{\langle \bar{d}^4 \rangle \square}{\partial_-^4}, \quad (\text{B.16})$$

where $\delta^{(12)}(z - z') = \delta^{(4)}(x - x')\delta^{(4)}(\theta - \theta')\delta^{(4)}(\bar{\theta} - \bar{\theta}')$. Then $\Delta(z, z')$ is defined by the condition

$$\int d^{12}z'' \Delta(z, z'') K(z'', z') = \delta^{(12)}(z - z'), \quad (\text{B.17})$$

or, introducing a chiral test superfield, $\Psi(z)$,

$$\int d^{12}z'' \int d^{12}z' \Delta(z, z'') K(z'', z') \Psi(z') = \Psi(z). \quad (\text{B.18})$$

Using the explicit form (B.16) of $K(z, z')$ we have

$$\begin{aligned} \Psi(z) &= \int d^{12}z' \int d^{12}z'' \Delta(z, z'') \delta^{(12)}(z'' - z') \left(-3 \frac{\langle \bar{d}^4 \rangle \square}{\partial_-^4} \Psi \right)(z') \\ &= \int d^{12}z' \Delta(z, z') \left(-3 \frac{\langle \bar{d}^4 \rangle \square}{\partial_-^4} \Psi \right)(z'). \end{aligned} \quad (\text{B.19})$$

The solution for $\Delta(z, z')$ is of the form

$$\Delta(z, z') = k \frac{\langle d^4 \rangle}{(x - x')^2} \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}'), \quad (\text{B.20})$$

with k a constant to be fixed. Substituting into the right hand side of (B.19) we get

$$\begin{aligned} &\int d^{12}z' k \frac{\langle d^4 \rangle}{(x - x')^2} \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}') \left(-3 \frac{\langle \bar{d}^4 \rangle \square}{\partial_-^4} \Psi \right)(z') \\ &= -3k \int d^{12}z' \square \frac{1}{(x - x')^2} \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}') \left(\frac{\langle d^4 \rangle \langle \bar{d}^4 \rangle}{\partial_-^4} \Psi \right)(z') \\ &= -3k(2\pi)^2 4(4!)^2 \int d^{12}z' \delta^{12}(z - z') \Psi(z') = -\frac{3k(4!)^3 (2\pi)^2}{2} \Psi(z), \end{aligned} \quad (\text{B.21})$$

where we used integration by parts and the relations

$$\square \frac{1}{(x - x')^2} = (2\pi)^2 \delta^{(4)}(x - x') \quad (\text{B.22})$$

and

$$\langle d^4 \rangle \langle \bar{d}^4 \rangle \Psi(z) = 4(4!)^2 \partial_-^4 \Psi(z). \quad (\text{B.23})$$

The latter is valid for a chiral superfield $\Psi(z)$. From (B.21) we read off the value of the constant k ,

$$k = -\frac{2}{(4!)^3 (2\pi)^2}. \quad (\text{B.24})$$

So the superfield propagator is

$$\Delta_b^a(z - z') = -\frac{2}{(4!)^3} \frac{\delta_b^a}{(2\pi)^2} \frac{1}{(x - x')^2} \langle d^4 \rangle \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}'). \quad (\text{B.25})$$

B.2 Relation to component field propagators

In order to verify that the superfield propagator constructed in the previous subsection contain the correct propagators for the individual fields in the $\mathcal{N} = 4$ multiplet we now re-derive the $\Delta(z - z')$ starting from the component expansion of $\Phi(z)$.

In the following it will be convenient to rewrite the $\mathcal{N} = 4$ superfield (2.5) as

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) = & e^{-\frac{i}{\sqrt{2}}\theta^m\bar{\theta}_m\partial_-} \left[-\frac{1}{\partial_-}A(x) - \frac{i}{\partial_-}\theta^m\bar{\lambda}_m(x) + \frac{i}{\sqrt{2}}\theta^m\theta^n\bar{\varphi}_{mn}(x) \right. \\ & \left. + \frac{\sqrt{2}}{6}\varepsilon_{mnpq}\theta^m\theta^n\theta^p\lambda^q(x) - \frac{1}{12}\varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q\partial_-\bar{A}(x) \right]. \end{aligned} \quad (\text{B.26})$$

The kinetic terms in the $\mathcal{N} = 4$ light-cone component action are

$$S_0 = \int d^4x \left[\bar{A}(x)\square A(x) + \frac{1}{2}\varphi_i(x)\square\varphi^i(x) - \frac{i}{\sqrt{2}}\bar{\lambda}_m(x)\frac{\square}{\partial_-}\lambda^m(x) \right], \quad (\text{B.27})$$

where the relation between the six real scalar fields φ^i , $i = 1, \dots, 6$ and the φ^{mn} 's, $m, n = 1, \dots, 4$ in (B.26) involves Clebsch-Gordan coefficients and it is given explicitly in (A.9).

From (B.27) we get the free propagators for the component fields,

$$(\Delta^{(A)})_b^a(x-y) = \langle \bar{A}^a(x) A_b(y) \rangle = \frac{1}{(2\pi)^2} \frac{\delta_b^a}{(x-y)^2} \quad (\text{B.28})$$

$$(\Delta^{(\varphi)})_b^{aj}(x-y) = \langle \varphi^{aj}(x) \varphi_b^j(y) \rangle = \frac{1}{(2\pi)^2} \frac{\delta^{ij}\delta_b^a}{(x-y)^2} \quad (\text{B.29})$$

$$\Rightarrow (\Delta^{(\varphi)})_{bmn}^{apq}(x-y) = \langle \bar{\varphi}_{amn}(x) \varphi_b^{pq}(y) \rangle = \frac{1}{(2\pi)^2} \frac{(\delta_m^q\delta_n^p - \delta_m^p\delta_n^q)\delta_b^a}{(x-y)^2}$$

$$(\Delta^{(\lambda)})_{bm}^{an}(x-y) = \langle \bar{\lambda}_m^a(x) \lambda_b^n(y) \rangle = \frac{i\sqrt{2}}{(2\pi)^2} \partial_- \frac{\delta_m^n\delta_b^a}{(x-y)^2} = \frac{i\sqrt{2}}{(2\pi)^2} \frac{\delta_m^n\delta_b^a(x^+ - y^+)}{(x-y)^4}. \quad (\text{B.30})$$

We can now consider the superfield two-point function,

$$\Delta_b^a(x, \theta, \bar{\theta}; x', \theta', \bar{\theta}') = \langle \Phi^a(x, \theta, \bar{\theta}) \Phi_b(x', \theta', \bar{\theta}') \rangle. \quad (\text{B.31})$$

Using (B.26), we expand this two-point function as

$$\begin{aligned} \langle \Phi^a(x, \theta, \bar{\theta}) \Phi_b(x', \theta', \bar{\theta}') \rangle = & e^{-\frac{i}{\sqrt{2}}(\theta^m\bar{\theta}_m\partial_- + \theta'^m\bar{\theta}'_m\partial'_-)} \langle \left[-\frac{1}{\partial_-}A^a(x) - \frac{i}{\partial_-}\theta^m\bar{\lambda}_m^a(x) \right. \\ & \left. + \frac{i}{\sqrt{2}}\theta^m\theta^n\bar{\varphi}_{mn}^a(x) + \frac{\sqrt{2}}{6}\varepsilon_{mnpq}\theta^m\theta^n\theta^p\lambda^{aq}(x) - \frac{1}{12}\varepsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q\partial_-\bar{A}^a(x) \right] \\ & \left[-\frac{1}{\partial'_-}A_b(x') - \frac{i}{\partial'_-}\theta'^r\bar{\lambda}_{br}(x') + \frac{i}{2\sqrt{2}}\varepsilon_{rsuv}\theta'^r\theta'^s\varphi_b^{uv}(x') + \frac{\sqrt{2}}{6}\varepsilon_{rsuv}\theta'^r\theta'^s\theta'^u\lambda_b^v(x') \right. \\ & \left. - \frac{1}{12}\varepsilon_{rsuv}\theta'^r\theta'^s\theta'^u\theta'^v\partial'_-\bar{A}_b(x') \right] \rangle, \end{aligned} \quad (\text{B.32})$$

where $\partial'_- = \partial/\partial x'^-$ and we used the reality condition

$$\bar{\varphi}_{mn}(x) = \frac{1}{2}\varepsilon_{mnpq}\varphi^{pq}(x) \quad (\text{B.33})$$

for the scalar field in the second superfield.

In the superspace two-point function (B.32) the only non-zero contractions correspond to the component two-point functions (B.28)-(B.30). Therefore we get

$$\begin{aligned} \langle \Phi^a(x, \theta, \bar{\theta}) \Phi_b(x', \theta', \bar{\theta}') \rangle &= e^{-\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m \partial_- + \theta'^m \bar{\theta}'_m \partial'_-)} \left[\frac{1}{12} \varepsilon_{mnpq} \theta^m \theta^n \theta^p \theta^q \langle \partial_- \bar{A}^a(x) \frac{1}{\partial'_-} A_b(x') \rangle \right. \\ &+ \frac{1}{12} \varepsilon_{mnpq} \theta'^m \theta'^n \theta'^p \theta'^q \langle \frac{1}{\partial_-} A^a(x) \partial'_- \bar{A}_b(x') \rangle - \frac{1}{4} \varepsilon_{mnpq} \theta^r \theta^s \theta'^m \theta'^n \langle \bar{\varphi}_{rs}^a(x) \varphi_b^{pq}(x') \rangle \\ &\left. - i \frac{\sqrt{2}}{6} \varepsilon_{mnpq} \theta^r \theta'^m \theta'^n \theta'^p \langle \frac{1}{\partial_-} \bar{\lambda}_r^a(x) \lambda_b^q(x') \rangle - i \frac{\sqrt{2}}{6} \varepsilon_{mnpq} \theta^m \theta^n \theta^p \theta'^r \langle \lambda^q{}^a(x) \frac{1}{\partial'_-} \bar{\lambda}_{r b}(x') \rangle \right], \end{aligned} \quad (\text{B.34})$$

Using (B.28)-(B.30) and integration by parts to get rid of the extra ∂_- 's, we find

$$\begin{aligned} \langle \Phi^a(x, \theta, \bar{\theta}) \Phi_b(x', \theta', \bar{\theta}') \rangle &= \delta_b^a e^{-\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m - \theta'^m \bar{\theta}'_m) \partial_-} \varepsilon_{mnpq} \left[-\frac{1}{12} \theta^m \theta^n \theta^p \theta^q - \frac{1}{12} \theta'^m \theta'^n \theta'^p \theta'^q \right. \\ &\quad \left. - \frac{1}{2} \theta^m \theta^n \theta'^p \theta'^q + \frac{1}{3} \theta^m \theta'^n \theta'^p \theta'^q + \frac{1}{3} \theta^m \theta^n \theta^p \theta'^q \right] \frac{1}{(2\pi)^2} \frac{1}{(x - x')^2} \\ &= -\frac{1}{12(2\pi)^2} \delta_b^a e^{-\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m - \theta'^m \bar{\theta}'_m) \partial_-} \frac{\delta^{(4)}(\theta - \theta')}{(x - x')^2}. \end{aligned} \quad (\text{B.35})$$

where we used the definition (A.8) of the fermionic δ -function. The super-propagator can be put in a more convenient form using the following identity

$$\langle d^4 \rangle \delta^{(4)}(\bar{\theta} - \bar{\theta}') = (4!)^2 e^{-\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m - \theta'^m \bar{\theta}'_m) \partial_-}, \quad (\text{B.36})$$

which can be proven expanding the left hand side as

$$\begin{aligned} \langle d^4 \rangle \delta^{(4)}(\bar{\theta} - \bar{\theta}') &= \varepsilon_{mnpq} \varepsilon^{rsuv} d^m d^n d^p d^q (\bar{\theta}_r - \bar{\theta}'_r)(\bar{\theta}_s - \bar{\theta}'_s)(\bar{\theta}_u - \bar{\theta}'_u)(\bar{\theta}_v - \bar{\theta}'_v) \\ &= (4!)^2 d^1 d^2 d^3 d^4 (\bar{\theta}_1 - \bar{\theta}'_1)(\bar{\theta}_2 - \bar{\theta}'_2)(\bar{\theta}_3 - \bar{\theta}'_3)(\bar{\theta}_4 - \bar{\theta}'_4) \end{aligned} \quad (\text{B.37})$$

and using (no sum over the repeated index k)

$$d^k(\bar{\theta}_k - \bar{\theta}'_k) = -1 + \frac{i}{\sqrt{2}}(\theta^k \bar{\theta}_k - \theta'^k \bar{\theta}'_k) \partial_- = -e^{-\frac{i}{\sqrt{2}}(\theta^k \bar{\theta}_k - \theta'^k \bar{\theta}'_k) \partial_-} \quad k = 1, \dots, 4. \quad (\text{B.38})$$

The identity (B.36) can be rewritten as

$$1 = \frac{1}{(4!)^2} e^{+\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m - \theta'^m \bar{\theta}'_m) \partial_-} \langle d^4 \rangle \delta^{(4)}(\bar{\theta} - \bar{\theta}'). \quad (\text{B.39})$$

Inserting (B.39) into the expression for the super-propagator we get

$$\begin{aligned} \langle \Phi^a(x, \theta, \bar{\theta}) \Phi_b(x', \theta', \bar{\theta}') \rangle &= -\frac{\delta_b^a}{12(2\pi)^2} e^{-\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m - \theta'^m \bar{\theta}'_m) \partial_-} \frac{\delta^{(4)}(\theta - \theta')}{(x - x')^2} \\ &\quad \times \frac{1}{(4!)^2} e^{+\frac{i}{\sqrt{2}}(\theta^m \bar{\theta}_m - \theta'^m \bar{\theta}'_m) \partial_-} \langle d^4 \rangle \delta^{(4)}(\bar{\theta} - \bar{\theta}'), \end{aligned} \quad (\text{B.40})$$

where we used the δ -function in (B.35) to change θ'^m into θ^m in the first exponential. The exponential factors in (B.40) cancel and we finally get

$$\langle \Phi^a(x, \theta, \bar{\theta}) \Phi_b(x', \theta', \bar{\theta}') \rangle = -\frac{2}{(4!)^3} \frac{\delta_b^a}{(2\pi)^2} \frac{\langle d^4 \rangle \delta^{(4)}(\theta - \theta') \delta^{(4)}(\bar{\theta} - \bar{\theta}')}{(x - x')^2}. \quad (\text{B.41})$$

in agreement with (B.25).

C Useful superspace relations

We collect in this appendix various relations used in manipulations of super Feynman diagrams in light-cone superspace.

Although $\frac{1}{\partial_-}$ is not a differential operator, it can be “integrated by parts” in superspace expressions. For generic superfields $f(x, \theta, \bar{\theta})$ and $g(x, \theta, \bar{\theta})$ we have

$$\begin{aligned} \int d^{12}z f(z) \frac{1}{\partial_-} g(z) &= \int d^{12}z \frac{\partial_-}{\partial_-} f(z) \frac{1}{\partial_-} g(z) \\ &= - \int d^{12}z \frac{1}{\partial_-} f(z) \frac{\partial_-}{\partial_-} g(z) = - \int d^{12}z \frac{1}{\partial_-} f(z) g(z). \end{aligned} \quad (C.1)$$

Using the definition (2.2) of the chiral derivatives, d^m and \bar{d}_m , and their commutation relation, it is easy to verify the following identity

$$\int d^{12}z_2 \delta^{(8)}(\theta_1 - \theta_2) \left[\langle d_{(1)}^4 \rangle \langle \bar{d}_{(1)}^4 \rangle \delta^{(8)}(\theta_1 - \theta_2) \right] = (4!)^4, \quad (C.2)$$

which is used repeatedly to carry out the integrations over the fermionic coordinates at each interaction vertex in superspace Feynman diagrams.

The commutation relation (2.3) for the superspace chiral derivatives implies

$$\overline{\langle \bar{d}^4 \rangle \langle d^4 \rangle \bar{d}_p \vec{d}_q} = 4! \varepsilon_{abpq} \overline{\partial_-^2 \langle \bar{d}^4 \rangle d^a \vec{d}^b}, \quad (C.3)$$

$$\overline{\langle \bar{d}^4 \rangle \langle d^4 \rangle \langle \bar{d}^4 \rangle} = 4(4!)^2 \overline{\partial_-^4 \langle \bar{d}^4 \rangle}. \quad (C.4)$$

The following identity can be verified using the normalisation of Grassmann integrals in appendix A

$$\int d^4\theta d^4\bar{\theta} \theta^m \theta^n \theta^p \theta^q \bar{\theta}_m \bar{\theta}_n \bar{\theta}_p \bar{\theta}_q = \frac{1}{4!}. \quad (C.5)$$

D Details of four-point function calculation

D.1 Diagrams involving cubic vertices

As pointed out in section 4.2.2 contributions to four-point functions of the Q^{ij} operators cannot be built using two cubic vertices of the same type (Vertex 3-I in (2.17) or Vertex 3-II in (2.18)). This can be seen from a simple counting of chiral derivatives and fermionic coordinates θ and $\bar{\theta}$.

We start by counting the superficial numbers (or powers) of d , \bar{d} , θ and $\bar{\theta}$ present in various factors used in constructing a four point function.

The superficial numbers (or powers) of various derivatives and fermionic variables in a four point function as shown in figure 6, are presented in table 2 for the three possible cases.

After performing the fermionic integrals in a super Feynman diagram, we are left with an *equal* number of θ 's and $\bar{\theta}$'s. Thus when fermionic coordinates are set to zero, a non-vanishing contribution can only arise if there are *equal* numbers of d 's and \bar{d} 's present to cancel the θ 's and $\bar{\theta}$'s. Thus, as can be seen from table 2, only the combination of one vertex of type 3-I and one of type 3-II can produce a non-zero result, as this is the only way of satisfying the above criterion. We illustrate this argument with the explicit example of a four point function at

Structure	d	\bar{d}	θ	$\bar{\theta}$
Propagator	4	0	4	4
Cubic Vertex 3-I	0	4	0	0
Cubic Vertex 3-II	0	8	0	0
External φ field in \mathcal{Q}	0	2	0	0

Table 1: Superficial powers of $d, \bar{d}, \theta, \bar{\theta}$

Combination of vertices	d	\bar{d}	θ	$\bar{\theta}$	$d\theta$	$d\bar{\theta}$
Vertex 3-I and Vertex 3-II	20	20	20	20	8	8
Vertex 3-I twice	20	16	20	20	8	8
Vertex 3-II twice	20	24	20	20	8	8

Table 2: Superficial powers of $d, \bar{d}, \theta, \bar{\theta}, d\theta, d\bar{\theta}$ in a four point function

one loop constructed using only the first cubic vertex. In this example, we focus on a specific contraction, where the legs labelled by colours b_5 and a_6 in figure 6 carry the $(\langle \bar{d}^4 \rangle / \partial_-^3)$ factor from the cubic vertex. We will henceforth suppress numerical factors, space-time derivatives and all tensor structures in the colour and flavour indices as they will not be important in the rest of the argument. The four point function evaluates to

$$\int d^4\theta_5 d^4\bar{\theta}_5 d^4x_5 \int d^4\theta_6 d^4\bar{\theta}_6 d^4x_6 \\ \times \left(\langle \bar{d}^4 \rangle \langle d^4 \rangle \frac{\delta_{51}^8}{x_{51}^2} \overleftarrow{d\bar{d}} \right) \left(\langle d^4 \rangle \frac{\delta_{52}^8}{x_{52}^2} \overleftarrow{d\bar{d}} \right) \left(\langle d^4 \rangle \frac{\delta_{56}^8}{x_{56}^2} \right) \left(\langle d^4 \rangle \frac{\delta_{64}^8}{x_{64}^2} \overleftarrow{d\bar{d}} \right) \left(\langle \bar{d}^4 \rangle \langle d^4 \rangle \frac{\delta_{63}^8}{x_{63}^2} \overleftarrow{d\bar{d}} \right) . \quad (\text{D.1})$$

Using the relation

$$\langle \bar{d}^4 \rangle \langle d^4 \rangle \delta^8 \overleftarrow{d\bar{d}} \sim \langle \bar{d}^4 \rangle \langle d^4 \rangle \bar{d}\bar{d} \delta^8 \sim \langle \bar{d}^4 \rangle d d \delta^8 ,$$

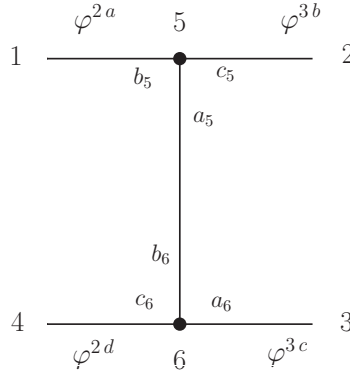


Figure 6

the integrand simplifies to

$$\left(\langle \bar{d}^4 \rangle d d \frac{\delta_{51}^8}{x_{51}^2}\right) \left(\langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{52}^8}{x_{52}^2}\right) \left(\langle d^4 \rangle \frac{\delta_{56}^8}{x_{56}^2}\right) \left(\langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{64}^8}{x_{64}^2}\right) \left(\langle \bar{d}^4 \rangle d d \frac{\delta_{63}^8}{x_{63}^2}\right).$$

We partially integrate $\langle d^4 \rangle$ from the second bracket to the first bracket, use the simplifying relation $\langle d^4 \rangle \langle \bar{d}^4 \rangle d d \delta^8 \sim \langle d^4 \rangle \bar{d} \bar{d} \delta^8$, and move the two \bar{d} 's in the second bracket to the first bracket as this is the only term that may survive once we perform all fermionic integrals and set θ 's and $\bar{\theta}$'s to zero. We first integrate over θ_5 and $\bar{\theta}_5$ using the free delta function δ_{52}^8 , and obtain

$$\begin{aligned} & \int d^4 x_5 \left(\bar{d} \bar{d} \langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{21}^8}{x_{51}^2} \right) \left(\frac{1}{x_{52}^2} \right) \\ & \times \int d^4 \theta_6 d^4 \bar{\theta}_6 d^4 x_6 \left(\langle d^4 \rangle \frac{\delta_{62}^8}{x_{56}^2} \right) \left(\langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{64}^8}{x_{64}^2} \right) \left(\langle \bar{d}^4 \rangle d d \frac{\delta_{63}^8}{x_{63}^2} \right). \end{aligned} \quad (D.2)$$

We partially integrate $\langle d^4 \rangle$ from the first bracket to the third bracket inside the fermionic integral to free up the delta function δ_{62}^8 , simplify the combination of the chiral and anti-chiral derivatives in the third bracket and then perform the remaining fermionic integrals to obtain,

$$\int d^4 x_5 \int d^4 x_6 \left(\bar{d} \bar{d} \langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{21}^8}{x_{51}^2} \right) \left(\frac{1}{x_{52}^2} \right) \left(\frac{1}{x_{56}^2} \right) \left(\langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{24}^8}{x_{64}^2} \right) \left(\langle d^4 \rangle \bar{d} \bar{d} \frac{\delta_{23}^8}{x_{63}^2} \right). \quad (D.3)$$

We find that the fourth and fifth brackets have an insufficient number of \bar{d} 's (precisely two each) to cancel the θ 's and thus this expression reduces to zero when we set the external θ 's and $\bar{\theta}$'s to zero.

If we choose to work with only the second cubic vertex, using similar manipulations we end up with an insufficient number of d 's (precisely two d 's per term in a total of two terms), so that the expression reduces to zero when we set the fermionic coordinates to zero.

Rule D.1 *In the topology shown in Figure 6, a cubic vertex cannot have component fields φ^i and φ^j with $i \neq j$, connected to any two of its legs.*

We will show why this is the case through an explicit calculation of a particular arrangement of the cubic vertices in figure 6 in which we assume that we have Vertex 3-I at point z_5 and Vertex 3-II at point z_6 . The calculations for the other permutations are identical. There are in total $3! \times 3!$ possible permutations of the cubic vertices with Vertex 3-I (2.17) at x_5 and Vertex 3-II (2.18) at x_6 . Thus a total of $3! \times 3! \times 2$ possible Wick contractions (including the cases when Vertex 3-I is at x_6 and Vertex 3-II at x_5). The contractions in figure 6 give

$$\begin{aligned} I &= \sigma^{2m_1 n_1} \sigma^{3m_2 n_2} \sigma^{3m_3 n_3} \sigma^{2m_4 n_4} \left(\frac{g^2}{12} \right) f^{a_5 b_5 c_5} f^{a_6 b_6 c_6} k^5 \\ & \times \int_{5,6} \left(\langle d^4 \rangle \frac{\delta_{51}^8}{x_{51}^2} \overleftarrow{d_{n_1} \bar{d}_{m_1}} \delta^{ab_5} \right) \left(\bar{\partial} \langle d^4 \rangle \frac{\delta_{52}^8}{x_{52}^2} \overleftarrow{d_{n_2} \bar{d}_{m_2}} \delta^{bc_5} \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^3} \langle d^4 \rangle \frac{\delta_{56}^8}{x_{56}^2} \frac{\overleftarrow{\langle \bar{d}^4 \rangle}}{\partial_-^2} \delta^{a_5 b_6} \right) \\ & \times \left(\frac{1}{\partial_-} \langle d^4 \rangle \frac{\delta_{63}^8}{x_{63}^2} \overleftarrow{d_{n_3} \bar{d}_{m_3}} \delta^{ca_6} \right) \left(\frac{\partial \langle \bar{d}^4 \rangle}{\partial_-^2} \langle d^4 \rangle \frac{\delta_{64}^8}{x_{64}^2} \overleftarrow{d_{n_4} \bar{d}_{m_4}} \delta^{dc_6} \right) \\ & = \sigma^{2m_1 n_1} \sigma^{3m_2 n_2} \sigma^{3m_3 n_3} \sigma^{2m_4 n_4} \left(\frac{g^2}{12} \right) f^{a_5 b_5 c_5} f^{a_6 b_6 c_6} k^5 \end{aligned} \quad (D.4)$$

$$\begin{aligned}
& \times \int_{5,6} \left(\langle d^4 \rangle \frac{\delta_{51}^8}{x_{51}^2} \overleftarrow{d_{n_1} \bar{d}_{m_1}} \delta^{ab_5} \right) \left(\bar{\partial} \langle d^4 \rangle \frac{\delta_{52}^8}{x_{52}^2} \overleftarrow{d_{n_2} \bar{d}_{m_2}} \delta^{bc_5} \right) \left(4(4!)^2 \partial_- \langle \bar{d}^4 \rangle \frac{\delta_{56}^8}{x_{56}^2} \frac{\overleftarrow{1}}{\partial_-^2} \delta^{a_5 b_6} \right) \\
& \times \left(\frac{1}{\partial_-} \langle d^4 \rangle \frac{\delta_{63}^8}{x_{63}^2} \overleftarrow{d_{n_3} \bar{d}_{m_3}} \delta^{ca_6} \right) \left(4! \varepsilon_{abn_4 m_4} \partial \langle \bar{d}^4 \rangle d^a d^b \frac{\delta_{64}^8}{x_{64}^2} \delta^{dc_6} \right), \tag{D.5}
\end{aligned}$$

where we used (C.4).

The second term inside the integral in (D.5) can be rewritten as $\left(\bar{\partial} \langle d^4 \rangle \bar{d}_{n_2} \bar{d}_{m_2} \frac{\delta_{52}^8}{x_{52}^2} \delta^{bc_5} \right)$, and now we can partially integrate $\langle d^4 \rangle$ entirely to the term containing δ_{56}^8 . Now, the derivatives \bar{d}_{n_2} and \bar{d}_{m_2} have two possible destinations on partial integration, but only when both move to the term containing δ_{51}^8 can we hope to get a non-zero contribution in the limit of fermionic coordinates going to zero. Thus I simplifies to

$$\begin{aligned}
& 4(4!)^3 \sigma^{2m_1 n_1} \sigma^{3m_2 n_2} \sigma^{3m_3 n_3} \sigma^{2m_4 n_4} \left(\frac{g^2}{12} \right) f^{a_5 b_5 c_5} f^{a_6 b_6 c_6} k^5 \delta^{ab_5} \delta^{bc_5} \delta^{a_5 b_6} \delta^{ca_6} \delta^{dc_6} \varepsilon_{rsn_4 m_4} \\
& \times \int_{5,6} \left(\bar{d}_{n_2} \bar{d}_{m_2} \langle d^4 \rangle \frac{\delta_{51}^8}{x_{51}^2} \overleftarrow{d_{n_1} \bar{d}_{m_1}} \right) \left(\bar{\partial} \frac{\delta_{52}^8}{x_{52}^2} \right) \left(\partial_- \langle d^4 \rangle \langle \bar{d}^4 \rangle \frac{\delta_{56}^8}{x_{56}^2} \frac{\overleftarrow{1}}{\partial_-^2} \right) \left(\frac{\langle d^4 \rangle}{\partial_-} \frac{\delta_{63}^8}{x_{63}^2} \overleftarrow{d_{n_3} \bar{d}_{m_3}} \right) \left(\partial \langle \bar{d}^4 \rangle d^r d^s \frac{\delta_{64}^8}{x_{64}^2} \right) \\
& = 4(4!)^3 \sigma^{2m_1 n_1} \sigma^{3m_2 n_2} \sigma^{3m_3 n_3} \sigma^{2m_4 n_4} \left(\frac{g^2}{12} \right) f^{a_5 b_5 c_5} f^{a_6 b_6 c_6} k^5 \delta^{ab_5} \delta^{bc_5} \delta^{a_5 b_6} \delta^{ca_6} \delta^{dc_6} \varepsilon_{rsn_4 m_4} \\
& \times \int d^4 x_5 \left(\bar{\partial} \frac{1}{x_{52}^2} \right) \left(\bar{d}_{n_2} \bar{d}_{m_2} \langle d^4 \rangle \frac{\delta_{21}^8}{x_{51}^2} \overleftarrow{d_{n_1} \bar{d}_{m_1}} \right) \int_6 \left(\partial_- \langle d^4 \rangle \langle \bar{d}^4 \rangle \frac{\delta_{26}^8}{x_{56}^2} \frac{\overleftarrow{1}}{\partial_-^2} \right) \left(\frac{\langle d^4 \rangle}{\partial_-} \frac{\delta_{63}^8}{x_{63}^2} \overleftarrow{d_{n_3} \bar{d}_{m_3}} \right) \left(\partial \langle \bar{d}^4 \rangle d^r d^s \frac{\delta_{64}^8}{x_{64}^2} \right) \\
& = 4(4!)^6 \sigma^{2m_1 n_1} \sigma^{3m_2 n_2} \sigma^{3m_3 n_3} \sigma^{2m_4 n_4} \left(\frac{g^2}{12} \right) f^{a_5 b_5 c_5} f^{a_6 b_6 c_6} k^5 \delta^{ab_5} \delta^{bc_5} \delta^{a_5 b_6} \delta^{ca_6} \delta^{dc_6} \varepsilon_{rsn_4 m_4} \\
& \times \varepsilon_{n_2 m_2 n_1 m_1} \int d^4 x_5 \left(\bar{\partial} \frac{1}{x_{52}^2} \right) \left(\frac{1}{x_{51}^2} \right) \times J,
\end{aligned}$$

where

$$J \equiv \lim_{\theta, \bar{\theta} \rightarrow 0} \int_6 \left(\partial_- \langle d^4 \rangle \langle \bar{d}^4 \rangle \frac{\delta_{26}^8}{x_{56}^2} \frac{\overleftarrow{1}}{\partial_-^2} \right) \left(\frac{\langle d^4 \rangle}{\partial_-} \frac{\delta_{63}^8}{x_{63}^2} \overleftarrow{d_{n_3} \bar{d}_{m_3}} \right) \left(\partial \langle \bar{d}^4 \rangle d^r d^s \frac{\delta_{64}^8}{x_{64}^2} \right).$$

In the limit $\theta, \bar{\theta} \rightarrow 0$ the above expression vanishes since $\sigma^{2m_1 n_1} \sigma^{3m_2 n_2} \varepsilon_{n_2 m_2 n_1 m_1} = 0$.

In general, attaching φ^i and φ^j to two legs of a cubic vertex will result in a factor of $\sigma^{i mn} \sigma^{j pq} \varepsilon_{mnpq} = 8\delta^{ij}$. Thus such an arrangement with $i \neq j$ does not contribute.

This result can be understood in terms of component fields. The only cubic vertices involving two scalar fields in the $\mathcal{N} = 4$ action – in any gauge, including the light-cone gauge – are the minimal coupling to the gauge field. Since the latter is a flavour singlet, the interaction cannot change the flavour index carried by the scalar field.

D.2 Diagrams involving quartic vertices

Rule D.2 *In the topology shown in Figure 7, component fields φ^i and φ^j with flavour $i \neq j$, cannot simultaneously attach to those legs of the quartic vertex which are both chiral fields, or both anti-chiral fields³.*

³Here we use the term “anti-chiral” field to refer to superfields associated with legs in a diagram carrying a $\langle \bar{d}^4 \rangle / \partial_-^2$ factor. These were originally $\bar{\Phi}$ ’s before use of (2.9).

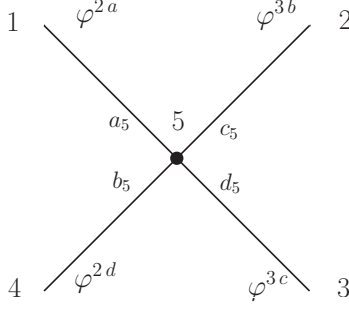


Figure 7

For a four point function constructed using Vertex 4-I (2.19), if the leg with colour index a_5 (chiral field) is connected with the external field φ^i , and the leg with colour index b_5 (chiral field) with the field φ^j , we get a factor of $\sigma^{imn}\sigma^{jpq}\varepsilon_{mnpq} = 8\delta^{ij}$ when evaluating the correlation function. The same thing happens with legs carrying colour indices c_5 and d_5 (anti-chiral fields) connected with external fields φ^i and φ^j . For Vertex 4-II (2.20), if the leg with colour index a_5 (chiral field) is connected with φ^i and the leg with index c_5 (chiral field) with φ^j , we get a factor of $\sigma^{imn}\sigma^{jpq}\varepsilon_{mnpq} = 8\delta^{ij}$. The same happens with legs carrying colour indices d_5 and b_5 (anti-chiral fields). Thus for such arrangements with $i \neq j$, the contraction vanishes.

This rule is verified by evaluating each permutation of the interaction vertex in Figure 7 and performing manipulations similar to those done in section D.1, for both Vertex 3-I and Vertex 3-II.

The only non-zero contributions to $G_4^{(H)}(x_1, \dots, x_4)$ at one loop come from diagrams involving a quartic vertex of type 4-II. As explained in section 4.2.3 there are various inequivalent Wick contractions to consider and we analyse them in detail below. We begin with

$$\begin{aligned}
 & \begin{array}{c} a_5 \quad b_5 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ d_5 \quad c_5 \end{array} \quad \equiv \quad V_4[a_5, b_5, c_5, d_5] \\
 & = \int_5 \delta^{aa_5} \delta^{bb_5} \delta^{cc_5} \delta^{dd_5} \left(\bar{d}_{p_1} \bar{d}_{q_1} \langle d^4 \rangle \frac{\delta_{15}^8}{x_{15}^2} \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \langle d^4 \rangle \frac{\delta_{52}^8}{x_{52}^2} \overleftarrow{d_{q_2} \bar{d}_{p_2}} \right) \\
 & \quad \times \left(\bar{d}_{p_3} \bar{d}_{q_3} \langle d^4 \rangle \frac{\delta_{35}^8}{x_{35}^2} \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \langle d^4 \rangle \frac{\delta_{54}^8}{x_{54}^2} \overleftarrow{d_{q_4} \bar{d}_{p_4}} \right) \\
 & \propto \varepsilon_{p_1 q_1 p_3 q_3} .
 \end{aligned}$$

Product with the common part $E_4[a_5, b_5, c_5, d_5]$ in (4.17) results in the contraction $\sigma^{2p_1 q_1} \sigma^{3p_3 q_3} \varepsilon_{p_1 q_1 p_3 q_3} = 0$. The reason why $V_4[a_5, b_5, c_5, d_5]$ leads to this contraction is explained under Rule D.2 above.

$$\begin{aligned}
& \begin{array}{c} a_5 \quad b_5 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ c_5 \quad d_5 \end{array} \quad \equiv V_4[a_5, b_5, d_5, c_5] \\
& = \int_5 \delta^{a_5 a} \delta^{b_5 b} \delta^{c_5 d} \delta^{d_5 c} \left(\bar{d}_{p_1} \bar{d}_{q_1} \langle d^4 \rangle \frac{\delta_{15}^8}{x_{15}^2} \right) \left(\frac{\langle \bar{d}^4 \rangle}{\partial_-^2} \langle d^4 \rangle \frac{\delta_{52}^8}{x_{52}^2} \overleftarrow{\bar{d}_{q_2} \bar{d}_{p_2}} \right) \left(\bar{d}_{p_3} \bar{d}_{q_3} \langle d^4 \rangle \frac{\delta_{35}^8}{x_{35}^2} \frac{\overleftarrow{\langle \bar{d}^4 \rangle}}{\partial_-^2} \right) \\
& \quad \times \left(\langle d^4 \rangle \frac{\delta_{54}^8}{x_{54}^2} \overleftarrow{\bar{d}_{q_4} \bar{d}_{p_4}} \right) \\
& = \int_5 \delta^{a_5 a} \delta^{b_5 b} \delta^{c_5 d} \delta^{d_5 c} \left(\langle d^4 \rangle \bar{d}_{q_1} \bar{d}_{p_1} \frac{\delta_{51}^8}{x_{51}^2} \right) \left(4! \varepsilon_{rsq_2 p_2} \langle \bar{d}^4 \rangle d^r d^s \frac{\delta_{52}^8}{x_{52}^2} \right) \left(4! \varepsilon_{uvq_3 p_3} \langle \bar{d}^4 \rangle d^u d^v \frac{\delta_{53}^8}{x_{53}^2} \right) \\
& \quad \times \left(\langle d^4 \rangle \bar{d}_{q_4} \bar{d}_{p_4} \frac{\delta_{54}^8}{x_{54}^2} \right), \tag{D.6}
\end{aligned}$$

where we used (C.3).

We now use the following rule for partially integrating $\langle d^4 \rangle$ to a product of two terms (disregarding the cases where both the terms are not acted upon by two d 's each),

$$\int d^4 \theta (\langle d^4 \rangle F)(GH) = 6 \varepsilon_{m_1 n_1 m_2 n_2} \int d^4 \theta F(d^{m_1} d^{n_1} G)(d^{m_2} d^{n_2} H), \tag{D.7}$$

and simplify $V_4[a_5, b_5, d_5, c_5]$ to

$$\begin{aligned}
& 6 \int_5 d^{a_5 a} \delta^{b_5 b} \delta^{c_5 d} \delta^{d_5 c} \left(\frac{\delta_{51}^8}{x_{51}^2} \right) \left(4! \varepsilon_{rsq_2 p_2} d^{m_1} d^{n_1} \langle \bar{d}^4 \rangle d^r d^s \frac{\delta_{52}^8}{x_{52}^2} \right) \\
& \quad \times \left(4! \varepsilon_{uvq_3 p_3} d^{m_2} d^{n_2} \langle \bar{d}^4 \rangle d^u d^v \frac{\delta_{53}^8}{x_{53}^2} \right) \left(\bar{d}_{q_1} \bar{d}_{p_1} \langle d^4 \rangle \bar{d}_{q_4} \bar{d}_{p_4} \frac{\delta_{54}^8}{x_{54}^2} \right) \varepsilon_{m_1 n_1 m_2 n_2}. \tag{D.8}
\end{aligned}$$

$V_4[a_5, b_5, d_5, c_5]$ as written in (D.8) simplifies to

$$\begin{aligned}
& 6(4!)^2 \int d^4 x_5 d^{a_5 a} \delta^{b_5 b} \delta^{c_5 d} \delta^{d_5 c} \left(\frac{\varepsilon_{m_1 n_1 m_2 n_2}}{x_{51}^2} \right) \left(\frac{(4!)^3 \varepsilon_{rsq_2 p_2} \varepsilon^{m_1 n_1 r s}}{x_{52}^2} \right) \\
& \quad \times \left(\frac{(4!)^3 \varepsilon_{uvq_3 p_3} \varepsilon^{m_2 n_2 uv}}{x_{53}^2} \right) \left(\frac{(4!)^3 \varepsilon_{q_1 p_1 q_4 p_4}}{x_{54}^2} \right), \tag{D.9}
\end{aligned}$$

in the limit $\theta, \bar{\theta} \rightarrow 0$. Using the following property of the Levi-Civita symbol

$$\varepsilon_{m_1 n_1 m_2 n_2} \varepsilon^{m_1 n_1 p q} = 2 (\delta_{m_2}^p \delta_{n_2}^q - \delta_{n_2}^p \delta_{m_2}^q), \tag{D.10}$$

we simplify

$$(\varepsilon_{m_1 n_1 m_2 n_2} \varepsilon^{m_1 n_1 r s}) \varepsilon_{rsq_2 p_2} (\varepsilon_{uvq_3 p_3} \varepsilon^{m_2 n_2 uv}) = 4 \varepsilon_{m_2 n_2 q_2 p_2} (\varepsilon_{uvq_3 p_3} \varepsilon^{m_2 n_2 uv}) = 16 \varepsilon_{p_3 q_3 p_2 q_2}. \tag{D.11}$$

Thus $V_4[a_5, b_5, d_5, c_5]$ (D.9) simplifies to

$$16 \times 6 \times (4!)^{11} \times d^{a_5 a} \delta^{b_5 b} \delta^{c_5 d} \delta^{d_5 c} \times \varepsilon_{p_3 q_3 p_2 q_2} \varepsilon_{p_1 q_1 p_4 q_4} \int d^4 x_5 \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}. \tag{D.12}$$

Substituting

$$k = (-1) \cdot 2 \cdot \frac{1}{(4!)^3} \frac{1}{(2\pi)^2}, \quad T(\sigma)_{\varepsilon\varepsilon\varepsilon\varepsilon} = 2^{12}, \quad (\text{D.13})$$

we obtain the final expression for $V_4[a_5, b_5, d_5, c_5]$ times the common part (4.18) as

$$-g^2 f^{eab} f^{eab} \frac{1}{(2\pi)^{12}} \frac{1}{x_{14}^2 x_{23}^2} \int d^4 x_5 \frac{1}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}. \quad (\text{D.14})$$

All permutations of the arguments in $V_4[a_5, b_5, d_5, c_5]$ of the form $[e_1, g_1, e_2, g_2]$ where $e_i \in \{a_5, c_5\}, g_i \in \{b_5, d_5\}$ or $e_i \in \{b_5, d_5\}, g_i \in \{a_5, c_5\}, i = 1, 2$, will have a non-zero contribution. The reason is explained under Rule D.2 above.

From the structure of Vertex 4-II (2.20), it is easy to see that

$$\begin{aligned} V_4[a_5, b_5, d_5, c_5] &= V_4[a_5, d_5, b_5, c_5] = V_4[c_5, b_5, d_5, a_5] = V_4[c_5, d_5, b_5, a_5] \\ &= V_4[b_5, a_5, c_5, d_5] = V_4[b_5, c_5, a_5, d_5] = V_4[d_5, a_5, c_5, d_5] = V_4[d_5, c_5, a_5, b_5]. \end{aligned}$$

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